1. [40 points] Short Answer (5 points each)
   
   (a) Let’s start with a question that virtually all of you got incorrect on the pre-test. Give the expression for the electric field, \( \vec{E} \) and the magnetic field, \( \vec{B} \) as a function of the scalar (\( \Phi \)) and vector (\( \vec{A} \)) potentials.
   
   \[ \vec{E} = -\nabla \Phi - \dot{\vec{A}} \]
   \[ \vec{B} = \nabla \times \vec{A} \]
   
   (b) The Gell-Mann matrices are the generators for the SU(3) group, the Lie group at the heart of the strong force. Two of the Gell-Mann matrices are:
   
   \[ \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \]
   
   What is \([\lambda_1, \lambda_4]??\)
   
   Sol.
   
   Admittedly, this is just a mathematical exercise, but I wanted to throw a few of them in there. Multiplying out:
   
   \[ \lambda_1 \lambda_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
   
   and
   
   \[ \lambda_4 \lambda_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]
   
   Subtracting yields:
   
   \[ [\lambda_1, \lambda_4] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \]
   
   In case you’re curious, this is just \( i \lambda_7 \).
   
   (c) In the previous part, if the two generators commute (or if they don’t) please describe in a sentence (or a diagram if you like) the implication with regards to the interactions of gluons.
   
   Sol.
   
   The very fact that these generators commute mean that the “free-field” Lagrangian associated with the vectors will have cross terms. Thus, there will be boson-boson interactions.
(d) Consider a *weak* force decay resulting in an antineutrino and an electron:

\[ X \to \bar{\nu}_e + e \]

Where \( X \) is an unknown spin-0 particle. (That is, the electron and neutrino have the same handedness and opposite momenta). Suppose further that the electron is emitted at a speed of 0.8. (You may also find it useful to make the electron momentum point in the z-direction).

Recalling that the weak force is left-handed, please determine how much this process is suppressed (e.g. how much slower the decay will occur) than if the weak force were ambidextrous.

As a reminder, left-handed Lagrangians contain terms like:

\[ \mathcal{L}_{int} = g X \bar{\nu}_e \gamma^\mu \left[ \frac{1}{2} (1 - \gamma^5) \right] e \]

As a further comment, don’t focus on the coupling constant. All you care about is the handedness.

**Sol.**

We’ve seen this before. The anti-neutrino is necessarily right-handed, so if the electron is going upwards, it must be spin-up (check your hands), and thus in \( u^+ \). The non-normalized version of \( u^+ \) is:

\[
 u^+ = \begin{pmatrix}
 \sqrt{\frac{m}{E+p}} \\
 0 \\
 \sqrt{\frac{E+p}{m}} \\
 0
 \end{pmatrix}
\]

Where we want to compute the relative probability of detecting the upper state (left-handed) versus the lower one. As a reminder:

\[
 \frac{m}{E+p} = \frac{1}{\gamma(1+v)}
\]

where this is the upper term squared. The lower term is simply the inverse. Thus:

\[
 P_L = \frac{1}{\frac{1}{\gamma(1+v)} + \gamma(1+v)} = \frac{1}{1 + \gamma^2(1+v)^2}
\]

\[
 = \frac{1}{1 + \frac{(1+v)^2}{(1-v)(1+v)}} = \frac{1}{1 + \frac{1+v}{1-v}} = \frac{1 - v}{2}
\]

Plugging in \( v = 0.8 \) we get:

\[ \text{suppression} = 10\% \]

(e) In a couple of sentences, explain how the “solar neutrino problem,” is explained by neutrino oscillation. Please be specific with regards to neutrino flavor.

**Sol.**

The sun emits electron neutrinos through the net process:

\[ 4H \to H e + 2e^+ + 2\nu_e \]

but the mass difference between the neutrinos and the mixing between flavor eigenstates causes some of the neutrinos to oscillate to \( \nu_\mu \). The detectors are only able to detect \( \nu_e \), and thus \( \sim 2/3 \) are missed.
(f) The Higgs mechanism was not originally part of the electroweak unification. What properties of the weak force was the Higgs mechanism meant to explain?

\textbf{Sol.}

The weakness of it. Without a massive $W^\pm$ and $Z^0$, the weak force and electromagnetism would have similar strengths. Actually, the weak force would be slightly stronger.

(g) What are the conservation laws associated with: i) Translational symmetry, ii) Rotational Symmetry, iii) SU(2) gauge invariance?

\textbf{Sol.}

i) Conservation of momentum.

ii) Conservation of angular momentum.

iii) Weak Isospin (I’ll also accept weak charge).

(h) In your own opinion, what are the 3 biggest unresolved issues in the Standard Model? A few words should suffice for each.

\textbf{Sol.}

There are lots, including:

i. Where handedness comes from.

ii. Why 3 generations?

iii. Too many free parameters.

iv. How to explain CP violations.

v. The electroweak-GUT “desert.”

vi. Why handedness?

vii. Why is charge quantized?

though I hope you won’t simply parrot back my own thoughts on this.

2. [30 points] Consider a Lagrangian of two interacting particles, $\phi$ (a real-valued scalar field), and $\psi$ (a \textit{massless} Dirac field), such that the complete Lagrangian of the universe is:

$$
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 + \overline{\psi}(i\gamma^\mu \partial_\mu)\psi - g\phi \overline{\psi}
$$

where $g$ is a dimensionless coupling constant (you can check if you like).

(a) Please write the Euler-Lagrange equation for $\overline{\psi}$.

\textbf{Sol.}

Easy enough:

$$
\overline{\psi}^\mu \partial_\mu \psi - g\phi \overline{\psi} = 0
$$

(b) Please draw the fundamental Feynman vertex for an interaction between the $\psi$ and $\phi$ fields. Be sure to label the strength of the vertex.

\textbf{Sol.}
(c) I’d like you to consider a decay of:

\[ \phi \rightarrow \psi + \bar{\psi} \]

where the \( \phi \) particle is at rest before the decay.

What is the momentum of the outgoing particles?

**Sol.**

Simple. Energy and momentum are conserved. The momenta of the \( \psi \) particles are each thus:

\[ p_\psi = \frac{m_\phi}{2} \]

(d) What is the decay rate of \( \phi \) particles? (\textbf{Hint:} For this part, there is a simplifying expression.)

**Sol.**

For two body decay, the solution is just:

\[ \Gamma = g^2 \frac{p}{8\pi m_\phi} = \left[ g^2 \frac{m_\phi}{32\pi} \right] \]

(e) (3 points) Suppose instead that two \( \psi \) beams are fired at each other with equal and opposite momenta and scatter off one another. Draw the lowest order Feynman diagram illustrating the process.

**Sol.**

This should be second nature to you by now, but if not, I wanted to limit the damage by only making this whole decay calculation worth 10 points.

\[ \begin{array}{c}
\psi_1' \\
\downarrow \\
g \\
\uparrow \\
\psi_2' \\
\downarrow \\
g \\
\uparrow \\
\psi_1 \\
\downarrow \\
\psi_2 \\
\end{array} \]

(f) (7 points) If the initial momenta of the particles, defined as \( p_I \), are much less than \( m_\phi \), the amplitude of the scatter can be found to be:

\[ A^2 = 2 \frac{g^4}{m_\phi^4} (1 - \cos \theta)^2 p_I^4 \]

where \( \theta \) is the angle that the \( \psi \) particles are deflected by the scatter.

Please compute \( d\sigma/d(\cos \theta) \).

As a helpful reminder:

\[ \int dE_1 \delta(\text{const.} - 2E_1) = \frac{1}{2} \]

**Please note:** This is the hardest part of the exam. It is only worth 7 points. No one will think the less of you if you move on and return to it later.

**Sol.**

First, note that:

\[ p_1 = \begin{pmatrix}
p_I \\
0 \\
0 \\
p_I \\
\end{pmatrix} \]
with $p_2$ having the reversed momentum.

Thus:

$$p_1 \cdot p_2 = 2p_1^2$$

The cross section can be written as:

$$\sigma = \frac{1}{16(2\pi)^2} \int A^2 \int \frac{1}{p_1 \cdot p_2} \frac{d^3p_1'}{E_1'} \frac{d^3p_2'}{E_2'} \delta(2p_1 - E_1') \delta(2p_1 - E_2') \delta(\vec{p}_1 + \vec{p}_2')$$

$$= \frac{1}{16(2\pi)^2} \frac{g^4}{m_\psi^2} \int (1 - \cos \theta)^2 \frac{d^3p_1'}{E_1'} \delta(2p_1 - E_1')$$

$$= \frac{1}{16(2\pi)^2} \frac{g^4}{m_\psi^2} \int (1 - \cos \theta)^2 \frac{2\pi E_1' dE_1'}{E_1^2} d(cos \theta) \delta(2p_1 - 2E_1')$$

$$= \frac{1}{64\pi} \frac{g^4}{m_\psi^2} \int (1 - \cos \theta)^2 d(cos \theta)$$

So:

$$\frac{d\sigma}{d\cos \theta} = \frac{1}{64\pi} \frac{g^4}{m_\psi^2} (1 - \cos \theta)^2$$

Note, by the way, that the dimensions are clearly correct. $[E]^{-2}$ for a cross-section.

3. [30 points] Suppose we have doublet of spinor fields:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

which have a weak hypercharge, $Y_W$. We will ignore handedness entirely for this problem.

The Lagrangian,

$$\mathcal{L} = \overline{\psi}_1 (i\gamma^\mu \partial_\mu) \psi_1 + \overline{\psi}_2 (i\gamma^\mu \partial_\mu) \psi_2$$

obeys a global (and local) $SU(2) \otimes U(1)$ symmetry under the transformation:

$$\left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \rightarrow e^{-i\frac{\theta^3}{2} Y_W} e^{-i\frac{\alpha}{2} \sigma^3} \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right)$$

(a) In a free field, what is the Euler-Lagrange equation associated with $\psi_2$?

**Sol.**

$$\partial_\mu (\overline{\psi}_2 (i\gamma^\mu)) = 0$$

(b) What are the conserved current(s) associated with $\alpha$ and $\theta^3$? Please leave the hypercharge, $Y_W$, and the coupling constants, $g'$ and $g_W$, in explicitly.

**Sol.**

What have done this many times. The conserved currents are:

$$J^{(0)\mu} = \frac{g'}{2} Y_W (\overline{\psi}_1 \gamma^\mu \psi_1 + \overline{\psi}_2 \gamma^\mu \psi_2)$$

and

$$J^{(3)\mu} = \frac{g_W}{2} (\overline{\psi}_1 \gamma^\mu \psi_1 - \overline{\psi}_2 \gamma^\mu \psi_2)$$

You’ll note that I’ve absorbed the $-$ sign. You can do so or not, as long as you do so consistently. I’ve done it so that the interaction Lagrangian has a negative.
(c) Assume a local gauge invariance, and label the two sets of vector fields \( B_\mu \) and \( W^{(1)}_\mu \). Please write out the Interaction Lagrangian between the Dirac fields and the \( W^{(3)}_\mu \) and \( B_\mu \) fields only.

**Sol.**

This, too, we've done a million times:

\[
\mathcal{L}_{\text{int}} = -J^{(0)} \cdot B - J^{(3)} \cdot W^{(3)}
\]

Though I'll find it useful to write this as:

\[
\mathcal{L}_{\text{int}} = -\left( \frac{g'}{2} Y_W B_\mu + \frac{g_W}{2} W^{(3)} \right) (\bar{\psi}_1 \gamma^\mu \psi_1) - \left( \frac{g'}{2} Y_W B_\mu - \frac{g_W}{2} W^{(3)} \right) (\bar{\psi}_2 \gamma^\mu \psi_2)
\]

(d) Now that we've introduced a current, what is the Euler-Lagrange equation associated with \( \psi_2 \)?

**Sol.**

Again, it’s fairly straightforward to show:

\[
\left[ i \partial_\mu - \frac{1}{2} \left( g' Y_W J^{(0)}_\mu - g_W J^{(3)}_\mu \right) \right] \gamma^\mu \psi_2 = 0
\]

(e) Suppose we are specifically looking at a doublet with \( Y_W = -1/3 \). Assume further that we rotate our vector fields:

\[
\begin{pmatrix}
B \\ W^{(3)}
\end{pmatrix} = \begin{pmatrix}
\cos \theta_W & \sin \theta_W \\
-\sin \theta_W & \cos \theta_W
\end{pmatrix} \begin{pmatrix}
A \\ Z
\end{pmatrix}
\]

such that

\[
g' = -g_W \tan \theta_W
\]

and

\[
q_e = g' \cos \theta_W
\]

Under these circumstances, what are the electric charges of \( \psi_1 \)?

**Sol.**

Let’s consider only the current terms in front of \( \psi_1 \):

\[
\left( \frac{g'}{2} Y_W B_\mu + \frac{g_W}{2} W^{(3)} \right) = \left( \frac{g'}{2} Y_W \cos \theta_W - \frac{g_W}{2} \sin \theta_W \right) A + \text{(blah)} Z
\]

\[
= -\frac{1}{2} g_W (Y_W \sin \theta_W + \sin \theta_W) A
\]

\[
= \frac{1}{2} q_e (Y_W + 1) A
\]

\[
= \frac{1}{3} q_e A
\]

We don’t care about the \( Z \) terms.

The charge is:

\[
Q_1 = \frac{1}{3}
\]

Similarly,

\[
Q_2 = -\frac{2}{3}
\]

(f) **E.C.** Given your previous answer, what particles could the doublet represent?

**Sol.**

They look for all the world like an anti-down and an anti-up, respectively.

(g) What are the masses of \( \psi_1 \) and \( \psi_2 \)? If this number differs from the measured physical masses of these particles, please comment on the difference in a sentence.

**Sol.**

Simply reading off the Lagrangian suggests that these are massless particles. The mass would be introduced by the Higgs mechanism.