PHYSICS T580: Standard Model Midterm Exam Solution Key (2016)

1. (20 points) Short Answer (5 points each)
   (a) As a reminder, the classical definition of angular momentum is:
   \[ \vec{l} = \vec{r} \times \vec{p} \]
   Based on this, what are the units of angular momentum in natural units?
   Sol.
   \[ [l] = [E]^0 \]
   You can get this directly by noting that spin has units of angular momentum in terms of \( \hbar \) and \( \hbar = 1 \). But also \( [r] = [E]^{-1} \) and \( [p] = [E]^1 \).
   (b) Let’s do a short essay question (1 or 2 sentences). How does the resolution of the negative energy positrons in classical Dirac theory relate to the Pauli Exclusion Principle?
   Sol.
   There’s a lot of flexibility in this answer, but I am essentially looking for three things:
   1) Treated as a classical theory, the positrons have a negative energy, but the same charge as the electron.
   2) Quantization of fermions involves postuating an ANTI-commutation relation for creation and annihilation operators, which reverses the charge and energy for the positron.
   3) That negative sign means that exchanges of particles are negatives of one another. Hence, \( |p, p\rangle = -|p, p\rangle \) for fermions, which is not possible unless the state is the vacuum.
   (c) Please give a short description of each of the “big three” discrete transformations, C, P, and T.
   Sol.
   C: Charge conjugation; reverse all electrical (and other quantum) charges
   P: Parity; reverse spatial coordinates
   T: Time inversion; reverse the arrow of time.
   (d) I’d like you to construct the simplest possible finite group that includes \( i \) under ordinary multiplication. Given the requirements for a group, list all possible members.
   E.C. Is this group Abellian?
   Sol.
   The group is Abellian, since order clearly doesn’t matter. The elements and multiplication table are:
   \[
   \begin{array}{c|ccccc}
   \circ & 1 & -1 & i & -i \\
   \hline
   1 & 1 & -1 & i & -i \\
   -1 & -1 & 1 & -i & i \\
   i & i & -i & -1 & 1 \\
   -i & -i & i & 1 & -1 \\
   \end{array}
   \]

2. (20 points) Consider a particle with the 4-momentum:
   \[
p^\mu = \begin{pmatrix}
   \frac{5}{2} \\
   0 \\
   0 \\
   \frac{3}{4}
   \end{pmatrix} \text{GeV}
   \]
   (a) What is the mass of the particle?
   Based on the approximate answer (within 10%), what type(s) of particle might this be?
   Sol.
The mass is simply:
\[ m^2 = E^2 - |\vec{p}|^2 = \left( \frac{25}{16} - \frac{9}{16} \right) \text{GeV}^2 = 1 \text{GeV}^2 \]
\[ m = 1 \text{GeV} \]

This is a proton or a neutron.

(b) Now boost the particle by \( v = 0.8 \) in the \( z \)-direction (the speed should increase). What is the 4-momentum in the boosted frame? Call it \( \tilde{p}^\mu \).

**Sol.**
First note:
\[ \gamma = \frac{5}{3} ; \ v\gamma = \frac{4}{3} \]
So the Boost is:
\[ \tilde{p}^\mu = \begin{pmatrix} 5/3 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4/3 & 0 & 0 & 3/4 \end{pmatrix} \begin{pmatrix} 5/4 \\ 0 \\ 0 \\ 3/4 \end{pmatrix} \text{GeV} = \begin{pmatrix} 37/12 \\ 0 \\ 0 \\ 35/12 \end{pmatrix} \text{GeV} \]

(c) What is the speed of the particle in the primed (boosted) frame?

**Sol.**
We calculate the speed via:
\[ v^3 = \frac{p^3}{p^0} \]
so
\[ v^3 = \frac{35}{37} \approx 0.946 \]

(d) Take a second vector:
\[ A^\mu = \begin{pmatrix} 8 \\ 0 \\ 0 \\ -4 \end{pmatrix} \]
as measured in the unboosted frame. What is \( p \cdot A \) as measured in the **boosted frame**?

**Sol.**
Tricked you! (Or hopefully not.)
The dot product doesn’t depend on frame so it’s way easier to do it in the unbarred frame:
\[ A \cdot p = \left( 8 \cdot \frac{5}{4} - (-4) \frac{3}{4} \right) = 13 \]
3. [25 points] Consider our old friend, the complex scalar field, but with an additional quartic potential:

\[ \mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* - \frac{1}{2} \lambda (\phi \phi^*)^2 \]

(a) Compute the Euler Lagrange equation for \( \phi^* \).

**Sol.**

First note:

\[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^*)} = \partial_\mu \phi \]

and

\[ \frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi - \phi^2 \phi^* \]

so:

\[ \partial_\mu \partial_\mu \phi = -m^2 \phi - \phi^2 \phi^* \]

(b) Compute the \( T^{11} \) component of the stress-energy tensor.

**Sol.**

First, note:

\[ T^{11} = \frac{\partial \mathcal{L}}{\partial (\partial_1 \phi_\alpha)} \partial_1 \phi_\alpha + \mathcal{L} \]

where I’ve used the fact that \( g^{11} = -1 \). Expanding out:

\[ T^{11} = 2 \partial^1 \phi \partial^1 \phi^* + \mathcal{L} \]

\[ = \dot{\phi} \dot{\phi}^* - \nabla \phi \cdot \nabla \phi^* + 2 \partial_1 \phi \partial_1 \phi^* - V(\phi) \]

\[ = [\dot{\phi} \dot{\phi}^* - \partial_2 \phi \partial_2 \phi^* - \partial_3 \phi \partial_3 \phi^* + \partial_1 \phi \partial_1 \phi^* - V(\phi)] \]

(c) For an isotropic field, \( T^{11} \) corresponds to the pressure. Isotropy means that:

\[ \partial_1 \phi \partial_1 \phi^* = \partial_2 \phi \partial_2 \phi^* = \partial_3 \phi \partial_3 \phi^* = \frac{1}{3} \nabla \phi \cdot \nabla \phi^* \]

What is the pressure in the isotropic limit?

**Note:** Your answer shouldn’t include individual spatial derivatives, only gradients.

**Sol.**

In that case:

\[ P = \dot{\phi} \dot{\phi}^* - \frac{1}{3} \nabla \phi \cdot \nabla \phi^* - V(\phi) \]

(d) The Lagrangian is invariant under a U(1) symmetry transformation:

\( \phi \rightarrow e^{-iq\theta} \phi ; \quad \phi^* \rightarrow e^{+iq\theta} \phi^* \)

What is the Noether current for this symmetry for our complex field?

**Sol.**

In some sense this is a trick, since the answer is identical to what we get for a complex field with any other potential. First note:

\[ \frac{\partial \phi}{\partial \theta} = -iq \phi \]

and

\[ \frac{\partial \phi^*}{\partial \theta} = iq \phi^* \]

and

\[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^*)} = \partial_\mu \phi \]

with a similar expression (complex conjugate) for \( \phi \). Combining:

\[ J^\mu = iq(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) \]
(e) We have seen that a complex scalar field may be decomposed into plane waves:

\[ \phi(x) = \int \frac{d^3p}{(2\pi)^2} \frac{1}{\sqrt{2E_p}} (b_p e^{-ip \cdot x} + c_p^* e^{ip \cdot x}) \]

with a quantization of the form:

\[ \hat{\phi} = \int (\ldots) (\hat{b} + \hat{c}^\dagger) \]

where \( \hat{b} \) annihilates a \( \phi \) particle, and \( \hat{c}^\dagger \) creates a \( \phi^* \) particle, and a transposed version for \( \hat{\phi}^\dagger \).

Imagine expanding out the quartic potential (interaction) term in the Lagrangian:

\[ \hat{L}_{int} = -\frac{1}{2} \hat{\phi} \hat{\phi}^\dagger \hat{\phi} \hat{\phi}^\dagger \]

In addition to the prefactors and explicit integrals (which you should ignore), the quantized version of the interaction potential will have combinations of 4 creation and annihilation terms.

Write at least 3 combinations of creation/annihilation terms arising from quantizing the field.

Assuming \( \phi \) particles have a charge of +1 and \( \phi^* \) have a charge of −1, what would be the net charge of these reactions?

**Note:** It seems like I’m asking a lot, but I just want quartets of possible operator combinations.

**Sol.**

First note the general expansion:

\[ \hat{L}_{int} = \lambda (\hat{b} + \hat{c}^\dagger)(\hat{b} + \hat{c}^\dagger)(\hat{b}^\dagger + \hat{c})(\hat{b}^\dagger + \hat{c}) \]

which yields:

\[ \lambda \hat{b} \hat{b}^\dagger \hat{b}^\dagger \]

+2 \( \phi \)-2\( \phi \): No change in charge

\[ \lambda \hat{b} \hat{b}^\dagger \hat{c} \]

+2\( \phi \)-1\( \phi \)+\( \phi^* \): No change in charge

\[ \hat{b} \hat{c}^\dagger \hat{c} \hat{b}^\dagger \]

+1\( \phi \)-1\( \phi \)+1\( \phi^* \)-1\( \phi^* \): No change in charge
4. [20 points] We have only occasionally encountered the group $\text{SO}(3)$, a group which is isomorphic to the more familiar $\text{SU}(2)$. Like $\text{SU}(2)$, $\text{SO}(3)$ has three generators:

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad X_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}; \quad X_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(a) What is the commutator, $[X_1, X_2]$?
If, indeed, our generators form a complete set, your answer should be a superposition of the generators above.

**Sol.**

$$[X_1, X_2] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = iX_3$$

(b) Compute the generalized form of $M_1(\theta)$, where the rotation is around the x-axis ($X_1$).
You may find the trigonometric expansions helpful:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - ...$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!} - ...$$

**Sol.**

We’ve done this before. We get:

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$

(c) Show that the matrix element, $M_1$, is orthogonal and special.

**Sol.**

This is intended to be simply a sanity check. The determinant is:

$$1 \cdot \cos \theta \cos \theta - (\sin \theta(-\sin \theta) \cdot 1) = 1$$

so it’s “Special.”

As for orthogonality:

$$M^{-1} = M^T$$

the lower 2x2 can be treated as its own matrix. The transpose simply reverses the sign on the sin, which is, of course, the inverse.

(d) The point of symmetries in field theories is that they leave Lagrangians invariant under transformations, including terms like $\Phi^T \Phi$. Consider a specific state for a triplet of real-valued scalar fields:

$$\Phi = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

For the specific case of a particle at rest:

$$Q_1 = i q \Phi^T X_1 \Phi$$

What is the $Q_1$ of the state above?
We begin with
\[ X\Phi = \begin{pmatrix} 0 \\ i \\ -i \end{pmatrix} \]
so
\[ Q_1 = iq\Phi^T X\Phi = -2q \]

5. [15 points] Consider a specific positive-energy, spin-up solution to the Dirac equation:

\[ u_+ = \frac{m}{\sqrt{E + p}} \begin{pmatrix} 1 \\ 0 \\ \frac{E + p}{m} \\ 0 \end{pmatrix} e^{-ip\cdot x} \]

where in this case, \( p \) refers specifically to the z-momentum of the particle.

(a) Compute and simplify \( \pi_+ \gamma^3 u_+ \).

**Sol.**

First, note:
\[ \gamma^0 \gamma^3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]
so
\[ \pi_+ \gamma^3 u_+ = N^2 \begin{pmatrix} 1 & 0 & \frac{E + p}{m} & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{E + p}{m} \\ 0 \end{pmatrix} \]
\[ = N^2 \left( \left( \frac{E + p}{m} \right)^2 - 1 \right) \]
\[ = \frac{m^2}{E + p} \left( \frac{E^2 + p^2 + 2pE - m^2}{m^2} \right) \]
\[ = \frac{1}{E + p} \left( 2p^2 + 2pE \right) \]
\[ = \frac{2p}{E + p} \]

(b) (10 points) We found that the Hamiltonian of a state is:

\[ \hat{H} = \gamma^0 (\gamma^i \partial_i + m) \]

Compute \( \hat{H} u_+ \).

You may find the process involved in the last part helpful.

**Sol.**

First, note:
\[ \partial_3 u_+ = -ip_3 u_+ = iju_+ \]

where the derivative is pulled down from the exponential.
So:

\[ \hat{H}u_+ = \gamma^0 (-i\gamma^3 \partial_3 + m)u_+ \]
\[ = (p\gamma^3 + mI)u_+ \]
\[ = \gamma^0 \begin{pmatrix} m & 0 & p & 0 \\ 0 & m & 0 & -p \\ -p & 0 & m & 0 \\ 0 & p & 0 & m \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{E+p}{m} \\ 0 \end{pmatrix} \]
\[ = \gamma^0 \begin{pmatrix} m + \frac{p(E+p)}{m} \\ 0 \\ -p + E + p \\ 0 \end{pmatrix} \]
\[ = \begin{pmatrix} E \\ 0 \\ \frac{m^2 + E^2 + p^2}{m} \\ 0 \end{pmatrix} \]
\[ = \begin{pmatrix} E \\ 0 \\ \frac{E^2 + Ep}{m} \\ 0 \end{pmatrix} \]
\[ = Eu_+ \]

as expected, where I ignored the normalization and the exponential.

(c) E.C.: 5 points

In class I noted that the probability of measuring a particle as right-handed versus left-handed is:

\[ P_R = \frac{\psi_3^2}{\psi_1^2 + \psi_3^2} \]

Recalling that \( p/E = v \), compute the probability as a function of \( v \). Simplify as much as possible.

Sol.

We have:

\[ P_R = \frac{(E+p)^2}{m^2} \frac{m^2}{m^2 + 1} \]
\[ = \frac{E^2 + 2p + p^2}{E^2 + 2pE + p^2 + m^2} \]
\[ = \frac{E^2 + 2p + p^2}{2E^2 + 2pE} \]
\[ = \frac{1 + 2v + v^2}{2(1 + v)} \]
\[ = \frac{(1 + v)^2}{2(1 + v)} \]
\[ = \frac{1 + v}{2} \]

Cool. \( P_R(v = -1) = 0 \) at \( P_R(v = 0) = 0.5 \), and \( P_R(v = 1) = 1 \).