Classical Mechanics I – HW # 2 Solution Key

HW 2 Solution Key

1. 3.3 (5 points)
   Sol.
   
   A shell traveling with velocity \( v \) explodes into three pieces of equal masses. One piece has velocity \( \vec{v}_1 = \vec{v}_0 \) and the other two have velocities that are equal in magnitude but mutually perpendicular. Find the two velocities and sketch the three velocities.

Defining \( 3m = M \), and \( v \) as speed of the two unknown the outgoing shells, we get:

\[
\vec{v}_2 = v \cos \theta_2 \hat{i} + v \sin \theta_2 \hat{j} \\
\vec{v}_3 = v \cos \theta_3 \hat{i} + v \sin \theta_3 \hat{j}
\]

The only way for momentum in the y-direction to be conserved is if \( \theta_2 = -\theta_3 \), so \( \sin \theta_3 = -\sin \theta_2 \) and \( \cos \theta_3 = \cos \theta_2 \). Further, for them to be perpendicular:

\[
\vec{v}_2 \cdot \vec{v}_3 = v^2 (\cos^2 \theta_2 - \sin^2 \theta_2) = 0
\]

which is satisfied for

\[
\theta = \pi/4 = 45^\circ
\]

Thus:

\[
v_2 = \frac{v_0 \hat{i}}{\sqrt{2}} + \frac{v_0 \hat{j}}{\sqrt{2}}
\]

2. 3.21 (5 points)

A uniform thin sheet of metal is cut in shape of a semicircle of radius \( R \) and lies in the xy plane with its center at the origin and diameter lying along the x axis. Find the position of the CM.

Sol.

Symmetry demands that the CM must lie along the y-axis. But where?

First, note that the total mass of a semicircle is:

\[
M = \sigma A = \frac{\pi R^2 \sigma}{2}
\]

where \( \sigma \) is the density per unit area.
Next:

\[ y_{CM} = \frac{\sigma \int y dA}{M} = \frac{\int \int (r \sin \phi) r \, dr \, d\phi}{M} = \frac{\sigma R^3 \sigma / 3 \int_0^\pi \sin \phi d\phi}{M} = \frac{-\sigma R^3 \sigma / 3 \cos \phi|_0^\pi}{M} = \frac{2\sigma R^3 / 3}{M} = \frac{4R}{3\pi} \]

3. 3.29 (5 points)

An uniform spherical asteroid of radius \( R_0 \) is spinning with angular velocity \( \omega_0 \). It picks up more matter until its radius is \( R \). The density remains the same and the additional matter was originally at rest. Find the final angular velocity if the radius doubles.

**Sol.**

Note a couple of things. First, angular momentum is conserved, so:

\[ \omega = \omega_0 \frac{I}{I_0} \]

Second, the moment of inertia of a sphere is:

\[ I = \frac{2}{5} MR^2 \]

If the radius doubles, \( M = 8M_0 \), and \( R^2 = 4R_0^2 \), so \( I = 32I_0 \). Thus:

\[ \omega = \frac{\omega_0}{32} \]

4. (15 points) A planet, of mass, \( m \) is orbiting a star of mass, \( M \). At some instant in its orbit, \( t_0 \), it is a distance, \( r_0 \) from the star, has a radial velocity, \( v_{r,0} \), a tangential velocity, \( v_{t,0} \).

**Note:** The algebra in the latter half of the problem is a bit hairy. You’ll be fine, though.

(a) In terms of the numbers above, compute the mechanical energy, \( T + U \), of the planet at \( t_0 \).

**Sol.**

The point of this problem is really to recognize that a few fixed numbers define the properties of an orbit entirely. In this case:

\[ T(t_0) = \frac{1}{2} m \left( v_{r,0}^2 + v_{t,0}^2 \right) \]

and

\[ U(t_0) = -\frac{GMm}{r_0} \]

These combine to produce a total (conserved) energy:

\[ E = \frac{1}{2} m \left( v_{r,0}^2 + v_{t,0}^2 \right) - \frac{GMm}{r_0} \]

a quantity we’ll use in later parts of the problem.
(b) In terms of the numbers above, compute the angular momentum, $L$ of the planet at $t_0$.

**Sol.**

For angular momentum, only the tangential velocity matters. Thus:

$$L = mv_{t,0}r_0$$

which is also conserved.

This means, for what it’s worth that at any point in the orbit:

$$v_t = \frac{L}{mr}$$

(c) There are two points in the orbit known as **periastron** and **apastron** that represent the nearest and closest approach to the star. At those times, the radial component of the velocity is zero. From conservation of mechanical energy and angular momentum, compute the distance from the star at apastron and periastron in terms of $M$, $r_0$, $v_{r,0}$, and $v_{t,0}$.

**Sol.**

You can, of course, leave all 5 variables in the expression if you like, but as we’ve already computed the conserved quantities, $E$, and $L$, I’m going to write out my solution in terms of those.

At periastron and apastron:

$$E = \frac{1}{2}mv_t^2 - \frac{GMm}{r}$$

with no radial term (and where $E$ is the conserved energy computed in part a). Substituting from the angular momentum calculation of $v_t$ gives:

$$E = \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

This is nice! There is only one unknown, $r$. It is even simpler if I multiply by $r^2$ and recast the equation as:

$$|E|r^2 - GMmr + \frac{L^2}{2m} = 0$$

where I cleaned up a little by noting that $E < 0$. At any rate, this is simply a quadratic equation, which has two solutions for $r$ (the smaller, given by the minus sign, is perihelion and the plus sign gives apastron.). So:

$$r_{a,p} = \frac{GMm \pm \sqrt{G^2M^2m^2 - 2|E|L^2/m}}{2|E|}$$

(d) What is the kinetic energy of the star at periastron and apastron? You may express your answer in terms of $r_{ast}$ and $r_{peri}$.

**Sol.**

In each case:

$$v_{a,p} = \frac{L}{mr_{a,p}}$$

so

$$T = \frac{1}{2}mv^2 = \frac{L^2}{2mr_{a,p}^2}$$

5. 4.4 (15 points)

_A particle of mass m is moving on a frictionless horizontal table and is attached to a massless string, whose other end passes through a hole in the table, where I am holding it. Initially the particle is moving in a circle of radius $r_0$ with angular velocity $\omega$, but I now pull the string down through the hole until a length $r$ remains between the hole and the particle._
(a) **What is the particles angular velocity now?**

From conservation of angular momentum we have:

\[ m\omega_0 r_0^2 = m\omega r^2 \]

so

\[ \omega = \omega_0 \frac{r_0^2}{r^2} \]

(b) **Assuming that I pull the string so slowly that we can approximate the particles path to be circle of slowly shrinking radius, calculate the work I did pulling the string.**

How much force are we applying? Well, the point of this question is that the force is the centripetal force, which is:

\[ F = \frac{mv^2}{r} = mr\omega^2 = \frac{mr_0^4\omega_0^2}{r^3} \]

where I’ve ignored the sign (the force is inward). The work done is thus:

\[ W = \int F \, dr = \int_r^{r_0} \frac{mr_0^4\omega_0^2}{r^3} \, dr = \frac{1}{2}mr_0^2\omega_0^2 \left( \frac{r_0^2}{r^2} - 1 \right) \]

where I cleaned up a bit in the last line.

(c) **Compare your answer to part (b) with the particles gain in kinetic energy.**

The kinetic energy at any given time is:

\[ T = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2 = \frac{1}{2}m\omega_0^2 \frac{r_0^4}{r^2} \]

Thus:

\[ \Delta T = \frac{1}{2}mr_0^2\omega_0^2 \left( \frac{r_0^2}{r^2} - 1 \right) \]

6. 4.8 (10 points)

A small frictionless puck perched at the top of a fixed sphere of radius R is given a tiny nudge and begins to slide down. Through what vertical height will it descend before it leaves the surface of the sphere?

**Sol.**

First, note that if the top of the sphere if \( \theta = 0 \), the puck will fall a vertical height, \( h \) such that:

\[ h = R(1 - \cos \theta) \]
as we did for the pendulum. But further, that descent gives us an increase in kinetic energy:

\[ \frac{1}{2}mv^2 = mgR(1 - \cos \theta) \]

or

\[ v^2 = 2gR(1 - \cos \theta) \]

To keep the sphere, the gravity must be sufficient to satisfy the centripetal force:

\[ mg \cos \theta = \frac{mv^2}{R} \]

\[ \cos \theta = 2(1 - \cos \theta) \]

so

\[ \cos \theta = \frac{2}{3} \]

\[ h = R(1 - \cos \theta) = \frac{R}{3} \]

7. 4.13 (5 points)

Calculate the gradient \( \nabla f \) of the following functions, \( f(x, y, z) \):

Sol.

(a) \( f = \ln(r) \) The problem was phrased rather awkwardly, but since we know how to take gradients in spherical coordinates, these are trivial:

\[ \nabla f = \frac{\partial f}{\partial r} \hat{r} = \frac{1}{r} \hat{r} \]

(b) \( f = r^n \)

\[ \nabla f = nr^{n-1} \hat{r} \]

(c) \( f = g(r) \)

\[ \nabla g = \frac{\partial g}{\partial r} \hat{r} \]

8. (20 points) Let's get a little practice taking divergence, gradient, and curl, since we're going to need to do these things later. Let's define a couple of scalar functions (one in Cartesian coordinates, and one in spherical) and a couple of vectors:

\[ f(x, y, z) = x^2 + y^2 \]

\[ g(r, \theta, \phi) = r \cos \theta \]

\[ \vec{u} = y\hat{i} - x\hat{j} \]

\[ \vec{v} = y\hat{i} + x\hat{j} \]

Please calculate:

(a) \( \vec{u} \times \vec{v} \)

Sol.

\[ \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ y & -x & 0 \\ y & x & 0 \end{vmatrix} = 2xy\hat{k} \]
(b) \( \nabla \times \vec{u} \)
I believe we did this one and the next one in class. It is:

**Sol.**

\[
\nabla \times \vec{u} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & x & 0 \\
\end{vmatrix}
\]

\[
= -2\hat{k}
\]

(c) \( \nabla \times \vec{v} \)

**Sol.**

\[
\nabla \times \vec{v} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & x & 0 \\
\end{vmatrix}
\]

\[
= 0
\]

(d) \( \nabla \cdot \vec{u} \)

**Sol.**

\[
\nabla \cdot \vec{u} = \frac{d(y)}{dx} + \frac{d(-x)}{dy} = 0
\]

(e) \( \nabla \cdot \vec{v} \)

Perhaps a bit redundant...

**Sol.**

\[
\nabla \cdot \vec{u} = \frac{d(y)}{dx} + \frac{d(x)}{dy} = 0
\]

(f) \( \nabla f \)

**Sol.**

\[
\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}
\]

\[
= 2x\hat{i} + 2y\hat{j}
\]

(g) \( \nabla g \)

**Sol.**

You can actually do this in 2 ways. The simplest is to note that \( g = x \), and thus:

\[
\nabla g = \frac{dg}{dx} \hat{i} = \hat{i}
\]

Picking the best coordinate system can be very useful.

But you can also use polar coordinates:

\[
\nabla g = \frac{\partial g}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial g}{\partial \theta} \hat{\theta}
\]
(Note that in 2-d, we typically used $\phi$ rather than $\theta$ in the class, but it'll work fine either way).

So:

$$\nabla g = \cos \theta \hat{r} - \sin \theta \hat{\theta}$$

I’ll leave it to you to show that this is the same as $\hat{i}$.

(h) Could either of $\vec{u}$ or $\vec{v}$ describe a conservative force (generated by $-\nabla U$)?

**Sol.**

Only $\vec{v}$ could, since it is curl free. Incidentally, the potential might take the form:

$$U_v = -xy + C$$

but you aren’t required to compute that.

9. (10 points) 4.30

A child toy has a shape of a cylinder mounted on top of a hemisphere. The radius of the hemisphere is $R$ and the CM of the whole toy is at a height $h$ above the floor.

**Sol.**

(a) Write down the gravitational potential energy when the toy is tipped to an angle $\theta$ from the vertical.

The center of mass, $h'$ and the angle, $\theta$ can be computed as:

$$h' = (h - R) \cos \theta + R = h \cos \theta + R(1 - \cos \theta)$$

Thus:

$$U = mgh' = mg(h \cos \theta + R(1 - \cos \theta))$$

relative to the floor. If we wanted to compute relative to vertical, we’d subtract $mgh$.

(b) Find the values of $R$ and $h$ for which the equilibrium at $\theta$ is stable.

First, note:

$$\frac{dU}{d\theta} = mg((R - h) \sin \theta)$$

which, naturally, is zero at $\theta = 0$ (the definition of an equilibrium). For it to be stable:

$$\frac{d^2U}{d\theta^2} > 0$$

so

$$\frac{d^2U}{d\theta^2} = mg((R - h) \cos \theta)$$

which is stable for $R > h$.

10. 4.53 (10 points)

**Sol.**

(a) Consider a electron in a circular orbit of radius $r$ around a fixed proton. The centripetal force is given by the Coulomb force. Prove that $T = -1/2U$, $E = T + U = -1/2U$.

All of these statements are, of course, equivalent. We actually did this in class. The force is:

$$|\vec{F}| = \frac{ke^2}{r^2} = \frac{mv^2}{r}$$
for a particle in a central potential and a circular orbit. The potential energy is:

\[ U = -\frac{k_e^2}{r} \]

(negative because it’s attractive.)

From the force equation, we get:

\[ mv^2 = 2T = \frac{k_e^2}{r} = -U \]

which proves the result.

(b) Consider the inelastic collision of an electron with a hydrogen atom. Write down the total energy of the system.

This is actually just a one-liner. Since the total energy of the bound atom is \( \frac{U}{2} \), then if \( T_1 \) is the energy of the electron prior to collision:

\[ E = \frac{U}{2} + T_1 \]

(c) Find out the final kinetic energy of electron 1, \( T'_1 \). Energy is conserved, so:

\[ T'_1 = T_1 + \frac{1}{2}U - \frac{1}{2}U'' \]

The electron can gain or lose energy by transferring it to the atom.