PHYSICS 432/532: Cosmology Final Exam Solution Key (2016)

1. [30 points] Short Answers (5 points each)

(a) The Bullet Cluster is at a redshift of $z = 0.351$. This isn’t quite small enough to use the “small redshift approximation,” $\chi = cz/H_0$, but for the purpose of the problem, please do so. What was the proper distance between us and the Bullet Cluster when the light we’re now seeing from it was emitted?

**Sol.**
First, note that a proper calculation of the comoving distance yields:

$$\chi = \frac{c}{H_0} \int_{1/(1+z)}^{1} \frac{da}{a^2 \sqrt{\frac{0.3}{a^2} + 0.7}} = 0.322 \frac{c}{H_0}$$

to be compared with:

$$\chi = \frac{cz}{H_0} = 0.351 \frac{c}{H_0}$$

about a 10% error.

The proper distance (based on the approximated comoving distance) is simply:

$$r = \frac{\chi}{(1 + z)} = 0.259 \frac{c}{H_0} = 779h^{-1}\text{Mpc}$$

For the “true” comoving distance it’s $0.238 \frac{c}{H_0}$.

(b) Using reasonable values, estimate the density contrast of a large galaxy cluster. You may find the critical density (found in the equation sheet) useful in computing the denominator.

**Sol.**
I am looking to see that you are able to estimate quantities on cosmological scales.
A large cluster has a mass of about $10^{15}M_{\odot} = 2 \times 10^{45}\text{kg}$, though a factor of 10 smaller is reasonable. The radius is:

$$R = 1.5h^{-1}\text{Mpc} = 6.6 \times 10^{22}\text{m}$$

(where I’ve taken $h = 0.7$) yielding a density of:

$$\rho = 1.65 \times 10^{-24}\frac{\text{kg}}{\text{m}^3}$$

The density of Dark Matter is:

$$\rho_M = \Omega_M \rho_c = 2.76 \times 10^{-27}\frac{\text{kg}}{\text{m}^3}$$

Thus:

$$\delta = \frac{\rho}{\rho_M} - 1 = 596$$

(c) In a sentence or two (perhaps employing an equation), explain why the primordial abundance of $^4\text{He}$ is approximately 25% by mass, regardless of parameters like $\Omega_B$.

**Sol.**
The ratio of neutrons to protons are set almost entirely by neutrinos decouple from baryons. That is, when processes like:

$$\nu_e + n \rightarrow p + e^-$$

become slow compared to the age of the universe, about 1 s after the big bang. At that time, the ratio of neutrons to protons becomes fixed at:

$$\frac{n_n}{n_p} = \exp \left( -\frac{\Delta mc^2}{k_BT} \right) \simeq \frac{1}{7}$$

For every 16 nucleons, we’ve got 2 neutrons, 14 protons. Virtually all of the neutrons end up in helium (along with 2 protons), giving 1 helium 12 protons, or 25% by mass.
(d) Below, I’ve sketched a relatively realistic rotation curve for a spiral galaxy, which increases linearly from \( R = 0 \) to \( R_c \), and then is outside of \( R_c \). I’ve also superimposed a thin line representing the light profile of the galaxy.

\[
\begin{align*}
\text{light} \\
v(R) \\
R
\end{align*}
\]

Based on the curve above, what is the density profile of the galaxy **interior** to \( R \leq R_c \) (the core) and **exterior** to \( R \geq R_c \) (the halo)? Express your answer as \( \rho \propto R^\alpha \).

**E.C.** (1 point) What does comparison with the light profile tell you about the presence or absence of dark matter in this galaxy?

**Sol.**

**Core:**

\[
v \propto R
\]

so

\[
v^2 \propto R^2 \propto \frac{M(R)}{R}
\]

and thus:

\[
M(R) \propto R^3
\]

Thus:

\[
\rho_{\text{core}} = \text{const}
\]

**Halo:**

\[
v \propto \text{const}
\]

so

\[
v^2 \propto \text{const} \propto \frac{M(R)}{R}
\]

so:

\[
M(R) \propto R
\]

so

\[
\rho_{\text{halo}} \propto \frac{1}{r^2}
\]

In any case, the mass outside of the core is substantial \((M \propto R)\), so there is lots of mass outside of where there is light. Lots of dark matter!
(e) What is the horizon problem and how does inflation “solve” it? Answer in a sentence or two, and be sure to address both parts.

**Sol.**
In non-inflationary universes, only small regions of space (∼ 100 comoving Mpc) are in causal or thermal contact with one another prior to recombination. This is a scale of about 1° on the sky as seen from earth today. However, the entire sky seems to be at the same temperature (up to tiny fluctuations) of about 2.73K. Inflation allowed thermal mixing on small scales which then inflated up to superhorizon scales (factors of ∼ 10^{100}).

(f) Compare (as ratios) the temperature, photon number density, and photon energy density at \( z = 1 \) to those values today.

**Sol.**
\( z = 1 \) corresponds to \( a = 1/2 \).
So:
\[
T = 2T_0 ; \ n_\gamma = 8n_\gamma(t_0) ; \ \rho_\gamma = 16\rho_\gamma(t_0)
\]
2. [10 points] Let us consider the growth of structure in a deSitter universe. As a reminder, this is one in which there is only a cosmological constant, $\Omega_{DE} = 1$ ($w = -1$), and a constant Hubble constant. (This is contrary to reality, since so far as we can tell, Dark Energy doesn’t cluster.)

Further, you may consider scales on which the pressure term in the structure growth equation are unimportant (that is, ignore them completely).

(a) Write down the simplified growth of structure differential equation for this bizarre universe.

**Sol.**

In this universe, we have:

$$\ddot{\delta} + 2H_0\dot{\delta} - \frac{3}{2}H_0^2\delta = 0$$

(b) Solve the growth of structure equation. You should get a result of the form: $\delta \propto e^{\alpha t}$. Solve for $\alpha$ explicitly.

**Sol.**

I’ve given you a big hint here. Taking the form above yields the quadratic relation:

$$\alpha^2 + 2H_0\alpha - \frac{3}{2}H_0^2 = 0$$

which gives two solutions:

$$\alpha_{\pm} = H_0\left(-1 \pm \frac{\sqrt{10}}{2}\right) \simeq 0.58H_0, -2.58H_0$$

(c) E.C. (3 points) Suppose we chose not to ignore the sound speed. Redo part (a) with the sound speed contribution included explicitly. (That is, what is the sound speed term for Dark energy?)

**Sol.**

It doesn’t actually get much more complicated. In this case, $c_s^2 = -c^2$, so:

$$\ddot{\delta} + 2H_0\dot{\delta} - \left(\frac{c_s^2k^2}{a^2} + \frac{3}{2}H_0^2\right)\delta = 0$$

3. [20 points] You are a cosmologist trying to determine whether you live in one of two possible universes:

- “A”: Radiation-Dominated ($\Omega_\gamma = 1$).
- “B”: A (nearly) empty universe ($\Omega_M = 0.01$, $\Omega_{\Lambda} = \Omega_{rad} \simeq 0$).

(a) What are the ages of the two universes in units of $1/H_0$? For universe “B” you may assume for calculation purposes that it’s completely empty.

**Sol.**

We’ve done this many times:

$$t_A = \frac{1}{2H_0}$$

$$t_B = \frac{1}{H_0}$$

(b) At what redshift would universe “B” go from being matter to curvature dominated?

**Sol.**

Recall:

$$H^2 = H_0^2 \left(\frac{\Omega_M}{a^3} + \frac{1 - \Omega_M}{a^2}\right)$$
which means that the two have equal contributions when:

\[ a = \frac{\Omega_M}{1 - \Omega_M} = 0.01 \frac{0.99}{0.99} = \frac{1}{1 + z} \]

Thus:

\[ 1 + z = 99 \]

or

\[ z = 98 \]

(c) What is the comoving horizon distance, in units of $c/H_0$ in each of these universes at the present epoch? For one of them, you may need to make an approximation.

**Sol.**

Recall:

\[ \chi_A = \frac{c}{H_0} \int_0^1 \frac{da}{a^2 \sqrt{\frac{1}{a^2}}} \]

\[ = \frac{c}{H_0} \]

For the open universe, there’s a problem, since:

\[ \chi_B = \frac{c}{H_0} \int_0^1 \frac{da}{a^2 \sqrt{\frac{1}{a^2}}} \]

\[ = \frac{c}{H_0} \ln(\frac{1}{0}) \rightarrow \infty \]

However, it’s clear that the at high redshifts (about 100 – see the previous part), that the universe becomes matter dominated, and converges quickly. Thus:

\[ \chi_B \sim \frac{c}{H_0} \ln(99) \]

(d) Suppose you discover a population of Type Ia SN at redshift $z = 1$. How much brighter (or dimmer) will the supernovae appear in universe A compared to universe B?

Note: your answer should be a dimensionless ratio, and you should make it clear which universe produces brighter supernovae.

Note # 2: For universe B, assume a completely empty model.

**Sol.**

In each case, we need to first compute the comoving distance to $a = 0.5$:

\[ \chi_A = \frac{1}{2} \frac{c}{H_0} \]

\[ \chi_B = \ln(2) \frac{c}{H_0} \simeq 0.693 \frac{c}{H_0} \]

In the latter case (a curved universe), we also need to know the radius of curvature, which is, conveniently:

\[ R_0 = \frac{c}{H_0} \]

so for universe B

\[ S_k(\chi) = R_0 \sinh(\chi/R_0) = 0.75 \frac{c}{H_0} \]
Exactly, incidentally.
The luminosity distance is defined as:

\[ D_L = (1 + z)S_k(\chi) \]

and thus:

\[ \frac{D_{L,A}}{D_{L,B}} = \frac{2}{3} \]

so

\[ \frac{f_A}{f_B} = \left( \frac{D_{L,B}}{D_{L,A}} \right)^2 = \frac{9}{4} \]

4. [15 points] I would like you to make a timeline (or timelist, if you prefer) of the big-bang, starting at the earliest time possible, and finishing with recombination (see? I’ve already given you a point on the timeline!). The events should be in proper order, and should include (at minimum) when electrons became non-relativistic, when equality occurred, when various fundamental forces split off from one another, and other major events.

You are not required to put exact times, expansion factors, temperatures, or energies at every step, but at least a couple will be required for full credit.

Also, this is not an essay. I don’t need you to describe the event. Listing them is more than sufficient.

**Sol.**

This timeline isn’t complete, but I’ve indicated with an asterisk those items which are required.

- * Planck Time: \(10^{43}\) sec.
- * Inflation/Decoupling of Strong Force/GUT Era - \(10^{35}\) s.
- Decoupling of Weak Force - \(10^{12}\) sec., \(E = 100\text{GeV} \).
- Quark Confinement - \(10^6\) s, \(E = 200-300\text{MeV} \)
- Muon Freeze-out.
- Pion Freeze-out.
- * Neutrino decoupling, P/N ratio fixed: 1 sec., \(E = 800\text{keV} \)
- Big Bang Nucleosynthesis ends: 3 min.
- Electrons become non-relativistic: \(E = 511\text{keV} \).
- * Equality: \((z=3900)\)
- * Recombination: \((z=1100)\)
5. [25 points] Here is a plot of the Planck power spectrum.

Along with three possible CMB maps (each approximately 10 degrees across, in case you’re curious).

(a) The 3 maps represent 3 different possible cosmologies in which $\Omega_M$ and $\Omega_\Lambda$ are allowed to vary including one which represents the concordance model.

Identify the geometry of the universe for the three maps (the geometric shapes for the three are all different), and tell me how that constrains either $\Omega_M$ or $\Omega_\Lambda$ individually, or in combination.

**Sol.**

Large angular structure corresponds to small angular diameter distance. The largest structure is in B. Thus:

- A: Flat ($\Omega_M + \Omega_\Lambda = 1$)
- B: Closed ($\Omega_M + \Omega_\Lambda > 1$)
- C: Open ($\Omega_M + \Omega_\Lambda < 1$)

(b) Suppose recombination occurred around $z = 1599$ (set as nearly realistic for the purpose of calculation simplification). Assuming that the universe is matter dominated during the entire history (it makes the calculation simpler, and is perfectly acceptable for what follows), what is the horizon scale at recombination (answers in $c/H_0$, please)?

**Sol.**

The horizon in an E-dS universe is:

$$\chi = \frac{c}{H_0} \int \frac{da}{a^2 \sqrt{\frac{1}{a^3}}} = 2\sqrt{\frac{a}{H_0}}$$

Thus, for $a = 1/1600$

$$\chi_{hor} = 0.05 \frac{c}{H_0}$$
(c) Assume the sound speed is a constant: \( c_s = 0.4c \). This significantly simplifies the calculation of the sound horizon. What is the wavenumber, \( k \) of the first acoustic peak in units of \( H_0/c \)?
You may leave in factors of \( \pi \).

**Sol.**
For the first acoustic peak:

\[ r_s k = \pi \]

In this case:

\[ r_s = 0.4\chi_{hor} = 0.02 \frac{c}{H_0} \]

so

\[ k = 50\pi \frac{H_0}{c} \]

(d) Compute the comoving distance to the surface of last scatter (you may set your lower limit to \( a = 0 \) for convenience) in the Einstein-deSitter universe, and from that, estimate the angular wavenumber, \( l \), of the first acoustic peak.

**Sol.**
For an E-dS universe:

\[ \chi = S_k = 2 \frac{c}{H_0} \]

and thus:

\[ l = S_k k = 100\pi \approx 314 \]

(e) **Qualitatively**, how would you expect the plot to change (but otherwise everything were held constant) if \( \Omega_b = 0.3 \)?
This should include not only the positions (values of \( l \)) of the peaks, but also the amplitudes of the peaks.

**Sol.**
- The sound speed would go down, so the sound horizon would shrink, and thus the peaks would all move to the right.
- The gravity of the baryons will contribute significantly to driving the waves, and thus the amplitudes will increase.