PHYSICS 432/532: Cosmology – Midterm Exam

February 11, 2014, 9:30am

You have until 11:00am to complete this exam. Please answer all questions clearly and completely. Make sure that you show all of your work. Only answers written in your bluebooks will be graded.

You may reference the formula sheet, attached. Beyond that, the exam is entirely closed book.

Calculators are not allowed on this exam. As such, you are expected to either leave expressions in purely analytic form or multiply out simple expressions like multiples of 10 (with units, of course). If you find yourself multiplying 3 digit numbers or taking the cube route of an integer explicitly, you're working too hard.

Finally, if you aren't sure how to get an answer in one part of the problem, pick a reasonable number (correct dimensions and order of magnitude) for subsequent parts of the problem. Propagated errors (provided they are done correctly) will not be penalized.

Undergrads: Answering grad-student questions can earn you 2 extra points each.

1. [25 *points*] Suppose we lived in a Dark Energy universe with $\Omega_{DE} = 1$ (and everything else set essentially to zero), with an equation of state of $w = -2/3$. **Note: This is not a cosmological constant.**

   (a) How does the density of dark energy in that universe scale with the expansion parameter?
   [6 points] Solution:
   $$\rho = \rho_0 a^{-3(1+w)} = \boxed{\rho_0 a^{-1}}$$

   (b) Does such a universe have a beginning? If so, what is the age of the universe at $a = 1$ in units of $1/H_0$?
   [6 points] Solution:
   $$t = \frac{1}{H_0}\int_0^1 \frac{da}{a\sqrt{\frac{1}{a}}} = \frac{2}{H_0}\sqrt{a}|_0^1 = \boxed{\frac{2}{H_0}}$$
   Yes, this universe does have a well-defined starting point.

   (c) Is this universe currently accelerating? Please compute $\ddot{a}$ at the present epoch. (As a reminder, at present, $\dot{a} = H_0$, so you may want to use that to make your expression look tidy).
   [7 points] Solution:
   The rate can be found by inverting the relationship from part b, and taking two time derivatives:
   $$\boxed{\ddot{a} = \frac{1}{2}H_0^2}$$
   Grad Students: Regardless of your answer, what is the maximum (least negative) value of w which produces an accelerating universe?
   Solution:
   Expressing the Hubble parameter as a general function of w:
   $$\frac{\dot{a}}{a} = H_0\sqrt{\Omega_{DE}\cdot\left[a^{3(w+1)}\right]}$$
   First, collect all factors of a to the right. Then, taking a derivative of this expression with respect to a gives us the following condition on w :
   $$-\frac{3}{2}w = \frac{1}{2}$$

Which means that for a universe which accelerates, the largest possible value of w should be:

$$\boxed{w = -\frac{1}{3}}$$

(d) Compute the comoving distance to a source at $z = 3$ in this universe. Express your answer as a ratio of c/H_0.
[6 points] Solution:

$$\chi = \frac{c}{H_0}\int_{\frac{1}{4}}^{1} \frac{da}{a^2\sqrt{\frac{1}{a}}} = \frac{c}{H_0} \cdot -2a^{-\frac{1}{2}} = \boxed{\frac{2c}{H_0}}$$

2. [25 *points*] Short Answer

(a) You observe a galaxy at redshift of $z = 0.01$. How far away is it in $h^{-1}Mpc$?
[6 points] Solution:

$$d = \frac{cz}{H_0} = \frac{3 \times 10^5 \cdot 0.01}{100h} = \boxed{30h^{-1}\text{ Mpc}}$$

Grad Students: Please estimate the fractional error on your calculation due to peculiar velocity.
Solution:
If said galaxy is part of a rich cluster, a typical velocity dispersion would be something like $\sigma = 900$ km/s. The redshift of an object with this peculiar velocity would be:

$$z = \frac{v}{c} = \frac{900}{3 \times 10^5} = 3 \times 10^{-3}$$

The error in distance would be the error in redshift, which turns out to be:

$$\frac{3 \times 10^{-3}}{0.01} = \boxed{0.3}$$

This is a 30% error in the distance, indicating that the redshift of a nearby galaxy is generally not the best tool available for measuring distances. The group or cluster itself would provide a better measurement of the distance.

(b) In a sentence or two each, describe the *two* best cosmological measurements of the Hubble Constant.
[6 points] Solution:

- **Supernovae Type Ia - Because these are considered to be *standard candles*, one can infer the rate of expansion of the universe by observing them at various distances (since the motion of these objects at cosmological distances are primarily dictated by the hubble flow).**
- **Strong lensing of quasars - Signals from multiple images of a background quasar can vary in brightness. If you can measure the time difference (known as a *time delay*) between these signals, one can get the Hubble Constant from purely geometrical arguments.**

(c) Describe in a couple of sentences (or perhaps also with an equation) the best observational evidence of a positive Cosmological Constant.
[6 points] Solution:
The important constraint is that the universe is consistent with being flat. If this is true, then we know that:

$$\Omega_{k,0} = 0 = 1 - \Omega_{m,0} - \Omega_{rel,0} - \Omega_\Lambda$$

Where it is necessarily true that $\Omega_{m,0} + \Omega_{rel,0} < 1$, forcing us to conclude that we do indeed get a positive cosmological constant. For what it's worth, the most accurate measurements are (errors omitted here):

$$\Omega_{m,0} = 0.32$$

$$\Omega_{\Lambda,0} = 0.68$$

(d) Suppose we lived in a matter-only universe with $\Omega_M = 2$. Such a universe would be destined for collapse. How much larger (in terms of a) would the universe get before it turned around?
[7 points] Solution:

$$\frac{\dot{a}}{a} = H_0\sqrt{\frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{k,0}}{a^2}}$$

$$\frac{2}{a^3} = \frac{1}{a^2} \rightarrow \boxed{a = 2}$$

3. [25 *points*] Consider a cluster as a standard ruler of fixed proper size, $l = 1h^{-1}Mpc$, placed at very redshift of $z = 8$ (much larger than a typical cluster) in either of 2 possible universes: a) Radiation-only ($\Omega_\gamma = 1.000001$), b) Einstein-deSitter.

(a) What is the value of Ω_K in each universe, and correspondingly, what is the shape and radius of curvature in each. Feel free to leave distances in units of c/H_0.
[6 points] Solution:

- **For a radiation-only universe:**

$$\Omega_k = -1.0 \times 10^{-6}$$

$$R_0 = \frac{c}{H_0\sqrt{|\Omega_k|}} = \frac{1000c}{H_0}$$

This universe would be considered to be closed.

- **For an EdS universe:**

$$\Omega_k = 0$$

$$R = \infty \text{ (also acceptable: undefined)}$$

This universe would be considered to be flat.

(b) What is the proper distance to the cluster in the E-dS Universe?
[6 points] Solution:

$$r = a\chi$$

$$\chi = \frac{c}{H_0}\int_{\frac{1}{9}}^{1} \frac{da}{a^2\sqrt{\frac{1}{a^3}}} = \frac{c}{H_0}2\sqrt{a}|_{\frac{1}{9}}^{1} = \frac{4}{3}\frac{c}{H_0}$$

$$\boxed{r = \frac{4}{27}\frac{c}{H_0}}$$

Why is $a \neq 1$ here? If it were, the proper distance would *always* be equal to the comoving distance to an object at any redshift, which cannot be true.

(c) What is the angular diameter distance to the cluster in the Einstein-deSitter Universe?
[6 points] Solution:

$$d_A = a \cdot \chi = \boxed{\frac{4}{27}\frac{c}{H_0}} = 444.4h^{-1} \text{ Mpc}$$

(d) In as simplified a form as you're comfortable (no need to do unit conversions to arcseconds), what is the angular size of the cluster in an Einstein-deSitter universe?
[7 points] Solution:

$$\theta = \frac{l}{d_A} = \frac{1h^{-1} \text{ Mpc}}{\frac{4}{27}3000h^{-1} \text{ Mpc}} = \boxed{\frac{27}{12000} \text{ radians}}$$

For completion of your notes (though you are not expected to have to calculate these), here is how many radians/arcseconds/arcminutes this subtends on the sky in such a universe:

$$\rightarrow 2.25 \times 10^{-3} \text{ radians} = 464.1 \text{ arcseconds} = 7.73 \text{ arcminutes}$$

Grad Students: Would the angular size be larger or smaller in the radiation-dominated universe? Please think carefully about whether the geometry or other considerations are more important. You need not give a final answer, but only a compelling argument.
Solution:
The intinsic size remains the same, so we should compare the angular diameter distance of the two universes.

$$d_A = a \cdot S_k(\chi)$$

For the radiation-only universe (closed), we need:

$$d_A = a \cdot R_0 \sin \frac{\chi}{R_0}$$

$$\chi = \frac{c}{H_0} \int_{\frac{1}{9}}^{1} \frac{da}{a^2 \sqrt{\frac{1}{a^4}}} = \frac{8}{9} \frac{c}{H_0}$$

We also know that R_0 will be large compared to the comoving distance to the cluster, so we may invoke the small angle approximation here.

$$d_A = \frac{8}{81} \frac{c}{H_0}$$

This fraction is smaller than the one in part c, which means the angle will be larger. A cluster in a radiation-only universe will appear to be larger than one in an EdS universe at this redshift.

4. [25 *points*] Let's think about a very early moment in the evolution of the universe, around a redshift of 10^{10}.

   (a) The temperature of the universe is currently 2.73K, but for this problem, you have my dispensation to be lazy, and say, $T_0 = 3K$.
   What was the approximate temperature of the universe at $z \simeq 10^{10}$?
   [5 points] Solution:
   $$T = \frac{T_0}{a} = \boxed{3 \times 10^{10} K}$$

   (b) What was the (approximate) energy of a typical photon at that epoch? Please round off the Boltzman constant to make the calculation tractable.
   [5 points] Solution:
   $$E_\gamma = k_B T = 1 \times 10^{-4} \frac{\text{ev}}{\text{K}} \cdot 3 \times 10^{10} \text{ K} = \boxed{3 \text{ Mev}}$$

   (c) What (if any) particles were relativistic at that time? Please list all of them.
   [5 points] Solution:
   - **Photons**
   - **Neutrinos**
   - **Electrons** (0.511 **Mev**)

   (d) Assuming all relativitstic species today are embodied in Ω_{rel} and assuming (correctly) that $z = 10^{10}$ is well inside the radiation-dominated epoch, please derive a general relationship algebraic relationship between time after the big bang and expansion factor. Your answer should include only a, H_0, and Ω_{rel}.
   [5 points] Solution:
   $$t = \frac{1}{H_0} \int \frac{a \cdot da}{\sqrt{\Omega_{rel}}} = \boxed{\frac{a^2}{2H_0\sqrt{\Omega_{rel}}}}$$

   (e) The currently density of protons is about $1m^{-3}$ *today.*
   As we discussed in class, the mean time between collisions for a particle interacting with a density, n of targets each with a cross-section, σ, and moving at relative speed, v is:
   $$\Gamma = n\sigma v$$
   For protons, the "Thomson Scattering Cross-Section" is approximately:
   $$\sigma = 10^{-35} m^2$$
   Approximately how long can a photon travel before striking a *proton* at $z = 10^{10}$?
   [5 points] Solution:
   $$n = \frac{n_0}{a^3} = 10^{30} m^{-3}$$
   $$\Gamma = 10^{30} \cdot 10^{-35} \cdot 3 \times 10^8 = 3 \times 10^3 s^{-1}$$
   So, about three thousand scatters per second. This means a photon can travel for one three-thousandth of a second before scattering off of a proton. In terms of distance, this is about 100 kilometers.

   (f) **Extra Credit** For electrons, the Thomson Scattering Cross-section is approximately a million times higher than for protons. At $z = 10^{10}$, what is the approximate mean travel time before a photon strikes an electron or a positron?
   Hint: Think carefully about the number density of electrons at this epoch.

Hint # 2: There number density of photons is approximately $10^9 m^{-3}$. You might find that useful.

Solution:

$$\Gamma = \left(10^{30} \cdot 10^9\right) \cdot 10^{-29} \cdot 3 \times 10^8 = 3 \times 10^{18} s^{-1}$$

So, it scatters 3×10^{18} times each second due to free electrons. In terms of distance, this is the size of an atom (1 $\mathring{A}$).