PHYSICS 432/532: Cosmology Midterm Exam Solution Key (2018)

1. [40 points] Short answer (8 points each)

(a) A galaxy is observed with a redshift of 0.02. How far away is the galaxy, and what is its lookback time?
   
   **Sol.**
   
   This is meant to simply be a low redshift calculation, so we can ignore which kind of distance we’re talking about here.
   
   \[d = \frac{cz}{H_0} = \frac{(3 \times 10^5 \text{ km/s})(0.02)}{100h \text{ km/s/Mpc}} = 60h^{-1} \text{ Mpc}\]

   The lookback time is simply:
   
   \[t_{\text{lookback}} = \frac{d}{c} = 195.6h^{-1} \text{ Myr}\]

   where I used the relation 1 pc = 3.26 ly.

(b) Consider a galaxy halo with a density of the form \(\rho \propto r^n\).
   
   i. Compute the coefficient, \(s\), in the rotation curve \(v \propto r^s\) for such a galaxy.
   
   **Sol.**
   
   Note that:
   
   \[M_R = \int 4\pi r^2 \rho(r)dr \propto R^{n+3}\]

   and

   \[v^2 = \frac{GM_R}{R} \propto R^{n+2}\]

   so

   \[s = \frac{n}{2} + 1\]

   ii. **(Required for Grad, +2 EC for Undergrad)** Now suppose the galaxy has a constant density out to some fixed radius, \(R\), and has zero density outside. Sketch the rotation curve.
   
   **Sol.**
   
   Inside \(R\), \(n = 0\), so, \(s = 1\) (linear increase). Outside, we have a Keplerian potential, so \(v \propto r^{-1/2}\). And, of course, the two curves must connect. Thus:

(c) Because I don’t want you to spend too much time on unit conversion, I’ll let you know that the Einstein radius for a galaxy of mass \(10^{12}h^{-1}M_\odot\) at a redshift of \(z = 0.1\) is about 5.2”.

Knowing that, what is the approximate Einstein radius (to within a factor of 3) of a massive galaxy cluster at a redshift of \(z = 0.3\)?

**Note:** To answer this, you need to have some reasonable idea of what the mass of a cluster is.
Sol.

First note that the Einstein radius is proportional to:

$$\theta_E \propto \sqrt{\frac{M}{D_d}}$$

A very massive galaxy cluster might be a few \( \times 10^{14} h^{-1} M_\odot \). So, taking 3=few (which should include all possible reasonable answers on your part), and noting that the distance increases by a factor of 10 (roughly) compared to an individual galaxy, we find that:

$$\theta_{E,\text{cluster}} = \sqrt{\frac{300}{3}} \theta_{E,\text{gal}} = 52''$$

You should find something in the range of a few tens of arc seconds to a few minutes.

(d) Please trace the following blank figure into your blue books.

Please mark the following:

i. Draw a large dot, •, to indicate the concordance model.

ii. Draw a dashed line to indicate all flat universes. On either side, indicate whether the universes are open or closed.

iii. Draw a solid line separating those universes which are accelerating from those which are decelerating.

iv. (Required for Grad, +1 EC for undergrad) I remind you that the age of the concordance model is (to within a few percent) \( \sim \frac{1}{H_0} \).

Knowing that, draw a line representing (approximately) all of the models with age \( t_0 = \frac{1}{H_0} \). Indicate which side of the line correspond to older universe, and which correspond to the younger ones.

Hint: You may need to think of a universe which also has an age equal to a Hubble time.

Sol.
Flat models have $\Omega_M + \Omega_\Lambda = 1$. The line separating accelerating from decelerating universes is given by:

$$\Omega_\Lambda = 2\Omega_M$$

The concordance model corresponds to $\Omega_M \simeq 0.3$, $\Omega_\Lambda \simeq 0.7$, and connecting that and the empty universe $t_0 = \frac{1}{H_0}$ gives the locus of constant ages. To the left, we have older universes, and to the right, with have younger. E.g. the deSitter Universe ($\Omega_\Lambda = 1$) is infinite, while E-dS is $2/3H_0$.

(e) To within a factor of 2, please give the current estimates of $\Omega_B$ and describe (in a sentence or two) why observationally at least some of the matter in the universe needs to be non-baryonic.

**Sol.**

$$\Omega_B \simeq 0.03$$

Any of the following are acceptable:

- The rotation curves of galaxies contain matter well beyond the luminous part.
- Cluster X-ray measurements suggest that the gas mass (from luminosity) is insufficient to account for the high temperature (which probes the potential).
- Cluster dynamics suggest missing mass (via the virial theorem).
- Lensing systems like the bullet cluster indicate that total mass doesn’t follow the gas.

2. [30 points] Consider a strange universe filled with “Pressureonium,” a substance with an equation of state of $w_P = 1$, such that $\Omega_P = 1$ (and with nothing else). The Hubble constant may be simply expressed as $H_0$.

(a) If the universe doubles in scale factor, by what factor will the density of Pressureonium increase or decrease?

**Sol.**

For $w = 1$,

$$\rho \propto a^{-3(1+w)} = a^{-6} = \frac{1}{64}$$

(b) In units of the Hubble time, how old is this universe?

**Sol.**
We've done this many times:

\[ dt = \frac{1}{H_0} \frac{da}{a \sqrt{\frac{\Omega_X}{a^2}}} \]

\[ = \frac{1}{H_0} \frac{da}{a \sqrt{\frac{1}{a^2}}} \]

\[ = \frac{1}{H_0} a^2 da \]

\[ t = \frac{1}{H_0} \int_{0}^{1} a^2 da \]

\[ = \frac{1}{3H_0} \]

(c) How does the expansion factor scale with time?

**Sol.**

Following the previous derivation, \( t \propto a^3 \) or

\[ a \propto t^{1/3} \]

(d) What is the horizon scale of this universe?

**Sol.**

We can do almost the exact same thing but with an extra factor of \( a \) downstairs:

\[ \chi = \int \frac{c}{H_0} \frac{da}{a^2 \sqrt{\frac{\Omega_X}{a^2}}} \]

\[ = \frac{c}{H_0} \int a \ da \]

\[ = \frac{c}{H_0} \frac{a^2}{2} \]

\[ \chi_{\text{hor}} = \frac{c}{2H_0} \]

3. [30 points] Consider an extremely overdense universe composed entirely of matter with \( \Omega_M = 5 \).

(a) What is the shape and (if not flat) the radius of curvature of this universe? (Express your answer in terms of the Hubble scale.)

**Sol.**

First, note that:

\[ \Omega_K = -4 \]

Thus, the universe is closed. So the radius of curvature is:

\[ R_0 = \frac{c}{H_0 \sqrt{|\Omega_K|}} = \frac{c}{2H_0} \]

(b) At what expansion factor will it collapse? If never, please justify.

**Sol.**

The Friedman equation includes:

\[ \left( \frac{\Omega_M}{a^3} + \frac{\Omega_K}{a^2} \right) \]
so we reach a turnaround point when:

\[ a_{\text{max}} = -\frac{\Omega_M}{\Omega_K} = 1.25 \]

(c) Now consider an observation of a galaxy with a radius of \( R = 1h^{-1}\) Mpc at a redshift of 1.

What is the angular size of the galaxy?

To assist you, I will note that the comoving distance is \( 0.422\frac{c}{H_0} \).

**Sol.**

First, note that:

\[
D_A = \frac{S_k(\chi)}{1 + z} = \frac{R_0 \sin(\chi/R_0)}{(1 + z)} = 0.187\frac{c}{H_0}
\]

so

\[
\theta = \frac{R}{D_A} = 0.00178 \text{ rad} = 368''
\]

(d) What is the ratio of surface brightness (flux/angular area) of the galaxy as seen at \( z = 1 \), compared to how it would look locally?

**Hint:** I promise you that this is a 1-line calculation. You may get the right answer another way, but you’re making life difficult for yourself.

**Sol.**

We’re looking for:

\[
b = \frac{f}{\theta^2} \propto \frac{D_A^2}{D_L^2} = (1 + z)^{-4} = \frac{1}{16}
\]

The surface brightness decreases by a factor of 16.