2013 Final Exam

You have 2 hours to complete the exam. Please answer all questions clearly and completely, and that you clearly indicate your final answer to each problem. Only material in your blue book will be graded.

You may use the formula sheet, attached. Beyond that, the exam is entirely closed book.

1. [25 *points*] Short answers (5 points each)
   (a) Let's start off conceptually. List (no detailed explanation is necessary) at least 3 observations or technologies which support GR.
   (b) A particle of mass, m is moving through flat spacetime at $v = 0.8$ in the x-direction. What is the 4-velocity of the particle?
   (c) In class, we found that Black Holes have an effective temperature:
   $$T \propto M^{-1}$$
   If the black holes behave like blackbodies, with a luminosity per unit area $\propto T^4$, how does the *lifetime* of a black hole scale with mass? (Forget, for a moment, about the complication that at present, most black holes are colder than the CMB).
   (d) A photon fluid has the property that $P = 1/3\rho$. Please write down the stress-energy tensor for a photon fluid at rest. Only ρ and definite numbers should appear in your answer.
   (e) A $2M_\odot$ neutron star has a radius of $6km$. How much slower does time run on the surface of a black hole compared to far away? As a reminder:
   $$M_\odot = 1.5km$$
   in geometrized units.
2. [20 *points*] The non-zero Christoffel symbols in a particular metric are:
   $$\begin{aligned} \Gamma^0_{01} = \Gamma^0_{10} &= -a \\ \Gamma^1_{00} &= -a \\ \Gamma^1_{11} &= a \end{aligned}$$
   While the metric is not specified, we do know that it is a function of the x-coordinate ($\mu = 1$) only. We will consider a region of space which looks locally like Minkowski space.

   The particle starts at rest.
   (a) Please list all conserved quantities for this metric. In particular, please note which if any of these terms are related to energy (and how).
   (b) Calculate the initial 4-velocity and the instantaneous acceleration ($\partial U^\alpha/\partial\tau$) on the particle using the geodesic equation.
   (c) At any artbitrary time, τ, write down the general geodesic equations in maximally simplifed form for all non-vanishing components of the 4-velocity. Make it explicit. There should be no dummy variables in your expression.

       This is essentially a set of coupled differential equations.
   (d) The particle travels a short enough distance such that, for the purposes of using the geodesic equation, the metric can be treated as nearly Minkowskian. After propert time, τ, what is the 4-velocity of the particle?

       Hint: Use some simplifying assumptions about relatively low speeds to make this computation. tractable.

       Hint #2: You may find it useful to solve for the x-component of the 4-velocity first.

(e) **E.C. (3 points)** Give a physical interpretation of both components of the 4-velocity (based on your answer to part d) in the non-relativistic limit. Why are the terms in the forms they are?

3. [30 *points*] Consider a 1+1 dimensional space, (t, x), with the metric:

$$g_{\mu,\nu} = \begin{pmatrix} -e^{kx} & 0 \\ 0 & 1 \end{pmatrix}$$

where k is a dimensional constant. You may even want to consider what the units of k must be.

Near the origin (x=0), this clearly produces the Minkowski metric.

(a) This metric has a stress-energy source which is (potentially) non-zero. Knowing nothing else, what is the scaling of the density, ρ in terms of k? (This is a dimensional analysis question, just in case you missed it).

(b) Compute all non-zero Christoffel symbols.

(c) A massive particle is instantaneously at rest at $x = 0$. What is the instantaneous acceleration on the particle?

(d) Compute the non-zero components of the Riemann tensor. You need not compute terms that are related to one another via the various symmetry relations. Please compute in the form $R^{\alpha}{}_{\beta\mu\nu}$.

(e) What are the non-zero terms in the Ricci Tensor and Ricci Scalar?

(f) What is the Einstein Tensor and the corresponding stress-energy tensor that gives rise to the metric. You may express both in the "downstairs" version, $G_{\mu\nu}$, $T_{\mu\nu}$.

4. [25 *points*] We've talked a lot about the Schwarzschild metric in this class. Consider an astronaut at rest at radius, $r_i > 2M$.

(a) What is the 4-velocity of the astronaut?

(b) The astronaut fires a photon of energy (according to her), E. The photon is fired radially outward toward $r = \infty$. What is the timelike component of the 4-momentum of the photon?

(c) What is the spacelike (radial) component of the 4-momentum?

(d) What energy does a distant observer measure for the photon? Compute the measured energy in the limit of:

$$\epsilon = r_i - 2M$$

is very small.

Remember, this is an important part of the calculation for Hawking Radiation.

(e) In class, we found that the critical reduced angular moment, $\tilde{L}$ for a circular orbit around a black hole was:

$$\tilde{L}^2 \equiv (u_\phi)^2 = \frac{Mr}{1 - 3M/r}$$

What is the reduced energy, $\tilde{E}$ for a circular orbit? (I don't expect you to have this memorized, but there are lots of ways to compute it.) Also, find the sign of $\tilde{E} - 1$ for arbitrary orbits $(\infty > r > 6M)$.