Homework # 3 Solution Key

1. You’re on your honor not to do this one by hand. I realize you can use a computer or simply look it up. Please don’t.

In a flat space, the metric in spherical coordinates, \( r, \theta, \phi \) is:

\[
 g_{\mu\nu} = \begin{pmatrix}
 1 & 0 & 0 \\
 0 & r^2 & 0 \\
 0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}
\]

(a) Please compute all non-zero Christoffel symbols for this system.

Sol.

First, you need to realize that the only non-zero derivatives are:

\[
 g_{\theta\theta,r} = 2r \\
 g_{\phi\phi,\theta} = 2r \sin^2 \theta \\
 g_{\phi\phi,\theta} = 2r^2 \sin \theta \cos \theta
\]

Since the metric is diagonal, we’re only to get terms with those three combinations of parameters. If the non-zero derivative is with respect to the \( r \) coordinate and \( r \) appears upstairs in the Christoffel symbol, we’re going to get a minus sign.

\[
 \Gamma^r_{\theta\theta} = -r \\
 \Gamma^r_{r\theta} = \frac{1}{r} \\
 \Gamma^r_{\phi\phi} = -r \sin^2 \theta \\
 \Gamma^\phi_{r\phi} = \frac{1}{r} \\
 \Gamma^\phi_{\phi\theta} = -\sin \theta \cos \theta \\
 \Gamma^{\phi}_{\theta\phi} = \cot \theta
\]

(b) Compute the divergence

\[
 V^\alpha_{,\alpha}
\]

Sol.

The general expansion is:

\[
 V^\alpha_{,\alpha} = V^\alpha_{,\alpha} + \Gamma^\alpha_{\alpha\beta} V^\beta
\]

So we only get to include those Christoffel symbols that have the same upstairs and downstairs components. This means:

\[
 V^\alpha_{,\alpha} = V^r_{,r} + V^\theta_{,\theta} + V^\phi_{,\phi} + \frac{2}{r} V^r + \cot \theta V^\theta
\]

If you like, this can be simplified to:

\[
 \nabla \cdot \vec{V} = \frac{1}{r^2} \frac{\partial r^2 V^r}{\partial r} + \frac{\partial V^\theta}{\partial \theta} + \cot \theta V^\theta + \frac{\partial V^\phi}{\partial \phi}
\]

Quick note: if you’re trying to make this look the same as the spherical divergence that you might find in the inside cover of Jackson, for example, you’re going to be disappointed. The components of a vector described in Jackson are normalized, but ours are not.
2. Consider a vector in 2-d flat space: \( \vec{v} = \hat{i} \) at the position \( \vec{r}_i = \hat{i} \)

(a) Express the vector and position in polar coordinates.

\textbf{Sol.}

This is pretty straightforward. Both are: \( \vec{v} = \vec{r} = \hat{r} \)

(b) Suppose you moved the vector to a position (in Cartesian coordinates):
\( \vec{r}_f = \hat{i} + 0.1\hat{j} \) without changing the vector itself. What would the components of the vector, \( \vec{v} \) be in polar coordinates at the final position?

\textbf{Method 1:} Use a straight geometric approach. Turn the Cartesian coordinates into polar coordinates.

\textbf{Sol.}

The radial component is simply computed via:
\[ V^r = \vec{V} \cdot \hat{r} = \frac{1}{\sqrt{1.01}} \]

and the \( \theta \) component via:
\[ V^\theta = 1 = \frac{1}{1.01} + 1.01(V^\theta)^2 \]

or
\[ V^\theta = -\left(\frac{0.01}{1.01}\right)^{1/2} \simeq -0.1 \]

so
\[ \vec{V}_f \simeq \hat{e}_r - 0.1\hat{e}_\theta \]

(c) \textbf{Method 2:} Use the relation:
\[ V^\alpha_{;\beta} = 0 \]

along the curve.

\textbf{Sol.}

I’m going to be more precise than is necessary here, and will save my approximations until the end. First, we need to write our differential equations:
\[
\begin{align*}
V^r_{,r} &= 0 \\
V^\theta_{,\theta} &= -\Gamma^\theta_{\phi\phi}V^\phi = rV^\phi \\
V^\phi_{,r} &= -\Gamma^\phi_{r\phi}V^\phi = -\frac{V^\phi}{r} \\
V^r_{,\theta} &= -\Gamma^r_{\phi\theta}V^\phi = -\frac{V^r}{r}
\end{align*}
\]

So, consider moving along a curve in the r-direction, a generic vector transforms as:
\[ V^r_2 = V^r_1 \]
\[ V_2^\theta = V_1^\theta \times \frac{r_1}{r_2} \]

(It’s a pretty straightforward differential equation).

Now, suppose we go along a curve of constant \( r \) but vary \( \theta \)? Generically, we get a coupled differential equation with the solution:

\[ V_3^r = V_2^r \cos(\delta \theta) + B \sin(\delta \theta) \]

And:

\[ V^\theta = \frac{1}{r} \frac{dV^r}{d\theta} = -\frac{V_2^r}{r} \sin(\delta \theta) + \frac{B}{r} \cos(\delta \theta) \]

Since for \( \delta \theta = 0 \), this must yield the original, we get:

\[ B = rV_2^\theta \]

and thus:

\[ V_3^r = V_2^r \cos(\delta \theta) + rV_2^\theta \sin(\delta \theta) \]

\[ V_3^\theta = V_2^\theta \cos(\delta \theta) - \frac{V_2^r}{r_2} \sin(\delta \theta) \]

\[ \frac{r_1}{r_2} V_1^\theta \cos(\delta \theta) - \frac{V_1^r}{r_1} \sin(\delta \theta) \]

This is exact. In our case, we’re moving from \( r_1 = 1 \) to \( r_2 = \sqrt{1.01} \), and from \( \theta = 0 \) to \( \theta = \sin \theta = \frac{0.1}{\sqrt{1.01}} \), and \( \cos \theta = \frac{1}{\sqrt{1.01}} \).

Plugging and chugging, we get:

\[ V^r = \frac{1}{\sqrt{1.01}} \]

and

\[ V^\theta = -\frac{0.1}{\sqrt{1.01}} \]

both of which are exactly what I got in the previous section.

3. 5.12

**Sol.**

We have the one-form with components:

\[ p_{\mu} = (x^2 + 3y, y^2 + 3x) \]

(a) \( p_{\alpha, \beta} \)

\[ p_{\alpha, \beta} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} \]
(b) Derivatives and then transform.

The transformation matrix is:

$$\Lambda^\alpha_{\beta} = \frac{\partial x^\alpha}{\partial x^\beta} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

But in order to solve for this, we first need to express the derivatives in terms of polar coordinates:

$$p_{\alpha,\beta} = \begin{pmatrix} 2r \cos \theta & 3 \\ 3 & 2r \sin \theta \end{pmatrix}$$

So, multiplying it out, we get:

$$p_{r,r} = 6 \sin \theta \cos \theta + 2r \sin^3 \theta + 2r \cos^3 \theta$$

$$p_{r,\theta} = p_{\theta,r} = -r(2r \sin \theta \cos^2 \theta - 6 \cos^2 \theta + 3 - 2r \cos \theta + 2r \cos^3 \theta)$$

and

$$p_{\theta,\theta} = 2r^2 \sin \theta \cos \theta (r \sin \theta + r \cos \theta - 3)$$

Hardly an elegant expression!

(c) $p_{\pi,\pi}$

This is a straightforward application, which first requires us to compute

$$p_{\pi} = \begin{pmatrix} r(6 \sin \theta \cos \theta + r \cos^3 \theta + r \sin^3 \theta) & -r^2 [r \sin \theta (\cos \theta - \sin \theta) + 3 - 6 \cos^2 \theta] \end{pmatrix}$$

From there, confirming the result from the previous part is straightforward. I could re-write it if you like, but I’m not going to.

4. 5.14

**Sol.**

Firstly, the non-zero derivatives of the tensor $A^{\mu\nu}$ are:

$$A^{rr}_{rr} = 2r$$
$$A^{r\theta}_{rr} = \sin \theta$$
$$A^{r\theta}_{r\theta} = r \cos \theta$$
$$A^{\theta r}_{rr} = \cos \theta$$
$$A^{\theta r}_{r\theta} = -r \sin \theta$$
$$A^{\theta \theta}_{rr} = \sec^2 \theta$$

Combined with the non-zero Christoffel symbols:

$$\Gamma^r_{\theta \theta} = -r ; \quad \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}$$
gives us the following in terms of Eq. 5.65:

\[
\begin{align*}
\nabla_r A_{rr} &= 2r \\
\nabla_r A_{r\theta} &= 2\sin \theta \\
\nabla_r A^{\theta r} &= 2\cos \theta \\
\nabla_r A^{\theta \theta} &= \frac{2}{r}\tan \theta \\
\n\nabla_\theta A_{rr} &= r^2(\sin \theta - \cos \theta) \\
\n\nabla_\theta A_{r\theta} &= r(1 + \cos \theta - \tan \theta) \\
\n\nabla_\theta A^{\theta r} &= r(1 - \sin \theta - \tan \theta) \\
\n\nabla_\theta A^{\theta \theta} &= \sec^2 \theta + \cos \theta + \sin \theta
\end{align*}
\]

5. 7.3

Sol.

The weak field metric:

\[
g_{\mu\nu} = \begin{pmatrix}
-(1 + 2\phi) & 0 & 0 & 0 \\
0 & 1 - 2\phi & 0 & 0 \\
0 & 0 & 1 - 2\phi & 0 \\
0 & 0 & 0 & 1 - 2\phi
\end{pmatrix}
\]

Since any derivatives are going to be 1st order in \( \phi \), the 1st order expansion of the Christoffel symbols can be approximated as:

\[
g^{\mu\nu} = \eta^{\mu\nu}
\]

Likewise, the only non-zero terms are:

\[
g_{\mu\nu,\lambda} = -2\delta_{\mu\nu}\phi_{,\lambda}
\]

In other words, we're going to get non-zero Christoffel symbols if two of the indices are the same. These will be of the following form:

\[
\Gamma^0_{i\theta} = -\phi_{,0} \\
\Gamma^0_{\theta i} = -\phi_{,0} \\
\Gamma^0_{00} = \phi_{,0} \\
\Gamma^0_{0i} = \phi_{,i} \\
\Gamma^0_{i0} = \phi_{,i} \\
\Gamma^0_{ij} = -\phi_{,j}
\]

6. A cosmic string is a theoretical construct (never observed in nature) which is infinitely long, and has a mass density per unit length of \( \lambda \). The coordinates describing the spacetime surrounding a cosmic string are:

\[
x^\mu = \begin{pmatrix}
t \\
R \\
\phi \\
z
\end{pmatrix}
\]

and which has a metric:

\[
g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & R^2(1 - 4\lambda) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
(a) Compute the volume element, $dV$, near the cosmic string.

**Sol.**
First, take the negative square root of the determinant:

$$g = -R^2(1 - 4\lambda)$$

so

$$dV = \sqrt{-gdx^0dx^1dx^2dx^3} = \sqrt{1 - 4\lambda RdRd\phi dzdt}$$

(b) Compute all non-zero Christoffel symbols. Please be smart, and start by noting that a) the metric is diagonal (for easy inversion), and b) most derivatives cancel.

**Sol.**
The only non-zero derivatives are:

$$g_{22,1} = 2R(1 - 4\lambda)$$

and thus, the only non-zero Christoffel symbols are:

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{R}$$

(c) Compute the distance between two points separated by $dx^\mu = dR$, and all other coordinates equal to zero. From that, compute the distance from the string itself out to a distance of $R = 1$.

**Sol.**
Very straightforward:

$$ds^2 = dR^2 \Rightarrow ds = dR$$

and thus:

$$s = \int_0^1 dR = 1$$

(d) Compute the distance between two points, each $R = 1$ from the string separated by an angle $d\phi$ (with all other $dx^\mu = 0$).

Using that, what is the total distance traversed by a particle covering a circular orbit of $R = 1$ around the cosmic string?

**Sol.**
Here, the distance is:

$$ds = \sqrt{1 - 4\lambda}d\phi$$

and thus, the distance around a circular orbit is:

$$s = 2\pi\sqrt{1 - 4\lambda}$$

(e) Compare parts c) and d) in the context of the normal relationship between radius and circumference. That is, does $C = 2\pi r$? If not, what should it be replaced with?

**Sol.**
This is odd. It is not $2\pi$ times the radius (unless $\lambda = 0$).