1.11.7

Sol.

A particle in a Schwarzschild metric (along the equator) has the dynamic relations:

\[
\left( \frac{dr}{d\tau} \right)^2 = \tilde{E}^2 - \left( 1 - \frac{2M}{r} \right) \left( 1 + \frac{\tilde{L}^2}{r^2} \right)
\]

Of course, if \( M = 0 \), this is simplified to:

\[
\left( \frac{dr}{d\tau} \right)^2 = \tilde{E}^2 - 1 - \frac{\tilde{L}^2}{r^2}
\]

which is 0 if:

\[
\frac{\tilde{L}^2}{r^2} = \tilde{E}^2 - 1
\]

or if

\[
r^2 = \frac{\tilde{L}^2}{\tilde{E}^2 - 1}
\]

Multiplying the top and bottom by \( m^2 \) gives a closest approach as:

\[
b^2 = r_{\text{min}}^2 = \frac{\tilde{L}^2}{\tilde{E}^2 - m^2}
\]

To show that the orbit is actually a straight line, we can relate the angle and the radius, \( r \). In other words, a simple parametric line would be:

\[
x = vt \quad ; \quad y = b
\]

or, in polar coordinates:

\[
\frac{x}{y} = \cot \phi
\]

so

\[
r^2 = b^2 + r^2 \cos^2 \phi
\]

or equivalently, a straight line is defined by:

\[
r = \frac{b}{\sin \phi}
\]

or we can further show:

\[
\frac{dr}{d\tau} = -b \frac{\cos \phi}{\sin^2 \phi} \frac{d\phi}{d\tau}
\]

or

\[
U^r = -\frac{r\sqrt{r^2 - b^2}}{b} U^\phi = -\frac{\sqrt{r^2 - b^2}}{rb} \tilde{L}
\]

Is this the case? We have the constraint:

\[
\vec{U} \cdot \vec{U} = -1
\]
or equivalently:

\[(U^r)^2 = \tilde{E}^2 - 1 - \frac{\tilde{L}^2}{r^2}\]

\[= \frac{\tilde{L}^2}{\tilde{b}^2} - \frac{\tilde{L}^2}{r^2}\]

\[= \frac{\tilde{L}^2}{r^2\tilde{b}^2}\]

so

\[U^r = \pm \tilde{L} \sqrt{\frac{r^2 - \tilde{b}^2}{rb}}\]

which is precisely what we found before for a straight line.

2. 11.21

**Sol.**

For a radial orbit, we have:

\[\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - 1 + \frac{2M}{r}\]

(a) To compute the proper infall time, we need:

\[\Delta \tau = \int_{r_i}^{r_f} \frac{dr}{\sqrt{\tilde{E}^2 - 1 + \frac{2M}{r}}}\]

Using \(\tilde{E} = 0.95\), and integrating from \(3M\) to \(2M\), we get:

\[\tau = 1.19M\]

(b) From \(r = 2M\) to \(r = 0\),

\[\tau = 1.374M\]

(c) There are only two non-zero 4-velocity components, \(U^0, U^r\). The first is easy to calculate:

\[U^0 = -g^{00}\tilde{E}\]

\[= \frac{0.95}{1 - 2M/2.001M}\]

\[= 1901\]

and

\[U^r = \sqrt{\tilde{E}^2 - 1 + \frac{2M}{r}} = 0.95\]

so:

\[U^\mu = \begin{pmatrix} 1901 \\ -0.95 \\ 0 \\ 0 \end{pmatrix}\]

(d) What about the photon that it sends out at that time?

A photon has a 4-momentum such that:

\[\vec{p} \cdot \vec{p} = 0\]
so:

\[ g_{rr}(p^r)^2 = -g^{00}E^2 \]

and thus:

\[ p^r = E \]

(which is fairly convenient)

What energy is the photon observed at locally?

\[
E_{\text{obs}} = -\vec{U} \cdot \vec{p} = -U_r p^r - U^0 p_0 = E(U^0 - U_r) = E(U^0 - U^r/(1 - 2M/r)) = 3800E
\]

so

\[
z = 3799
\]

Using the relations that we derived in class:

\[
a_y - \text{stretching} = \frac{2M}{r^3} \Delta y
\]

and

\[
a_x - \text{compressing} = \frac{M}{r^3} \Delta x
\]

Throughout this problem, assume that you dropped from rest at infinity.

(a) Find the smallest black hole in which you could survive long enough to pass the event horizon. You will need to do a little searching to determine the stresses that humans can survive.

**Sol.**

Pilots can handle around 10 g’s according to the good folks at Wikipedia. So we simply need to solve for:

\[
g_{\text{max}} = \frac{M}{(2M)^3} \Delta x
\]

or

\[
M = \frac{c^3}{G} \sqrt{\frac{\Delta x}{4g_{\text{max}}}}
\]

where I’ve helpfully put in the unit factors. Plugging in, we get:

\[
M = 2.8 \times 10^{34} kg = 1.4 \times 10^4 M_\odot
\]

(b) For a 1M_\odot black hole, how long does it take between the time you feel mildly uncomfortable (tidal force between head and feet is 2g) and you die? This should be in proper time, of course.

**Sol.**

As we’ve seen, the infall time from point “A” to point “B” is:

\[
\Delta \tau = \int_{r_1}^{r_f} \frac{dr}{\sqrt{\frac{2M}{r}}}
\]

where I’ve taken the liberty of assuming that \( \dot{E} = 1 \). Integrating, we get:

\[
\Delta \tau = \frac{2}{3} \sqrt{\frac{\frac{r^3}{2GM}}{r_2}}
\]
We just need to compute the appropriate radius for any given tidal force:

\[
r = \left( \frac{2GM\Delta g}{g} \right)^{1/3}
\]

So for a 1\(M_\odot\) black hole, we become uncomfortable at: \(3 \times 10^6m\), and we die at \(1.76 \times 10^6m\). The time it takes to die is:

\[
\Delta \tau = 0.19s
\]

(c) How about a 10\(M_\odot\) black hole?

**Sol.**

We now get uncomfortable around \(6.5 \times 10^6m\), and die at \(3.79 \times 10^6m\). Interestingly, the interval is almost exactly the same:

\[
\Delta \tau = 0.19s
\]

4. 12.9 - It is vital to this problem that your starting point be the Einstein field equation, and that you use an arbitrary non-zero density at the present.

**Sol.**

(a) Show that a photon which propagates on a radial null geodesic of the metric:

\[
g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & R^2(t) & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}
\]

has energy \(-p_0\) inversely proportional to \(R(t)\).

First, define some terms. To be a radial null geodesic, the momentum needs to be only along the radial direction, and needs to be lightlike:

\[
p \cdot p = 0
\]

so:

\[
g^{00}(p_0)^2 + g^{11}(p_1)^2 = 0
\]

which yields:

\[
p_0 = \frac{\sqrt{1 - kr^2}}{R(t)} p_1
\]

Of course, we don’t immediately have a value for \(p_1\), and it’s clear that it’s not a conserved quantity. Why? Because the metric is a function of \(r\). That’s why.

But to actually figure out the evolution, we’re going to need to compute a trajectory using Christoffel symbols. Notably, the only ones including only time and \(r\) are:

\[
\Gamma^0_{rr} = \dot{R} \frac{R}{1 - kr^2}
\]

\[
\Gamma^r_{0r} = \frac{\dot{R}}{R}
\]

\[
\Gamma^r_{rr} = \frac{kr}{1 - kr^2}
\]

As a reminder, the geodesic equation for photons is:

\[
\ddot{p}^\mu = -\Gamma^\mu_{\alpha\beta} \frac{p^\alpha p^\beta}{p^0}
\]
This yields two equations:
\[ \dot{p}^0 = -\Gamma_{rr}^0 \frac{p^r p^r}{p^0} \]

But because of the restrictions on a null geodesic:
\[ (p^r)^2 g_{rr} = -(p^0) g_{00} \]
or
\[ (p^r)^2 = \frac{(p^0)^2 (1 - kr^2)}{R(t)^2} \]
and so:
\[ \dot{p}^0 = -\left( \frac{\dot{R}}{R} \frac{R}{1 - kr^2} \right) \frac{1 - kr^2}{R(t)^2} p^0 \]
Thus:
\[ \dot{p}^0 = -\frac{\dot{R}}{R} p^0 \]
which is a simple diff eq. It is:
\[ p^0 \propto R^{-1} \]
Since an index-lowering of the timelike component simply adds a negative sign, it’s clear:
\[ p^0 \propto -\frac{1}{R} \]

(b) Show from this that a photon emitted at time \( t_e \) and received at time \( t_r \) by observers at rest in the cosmological reference frame is redshifted by:
\[ 1 + z = \frac{R(t_r)}{R(t_e)} \]

An observer at rest has \( U^i = 0 \), which means:
\[ (U^0)^0 = -1 \]
given the metric, this means that all stationary observers have:
\[ U^0 = 1 \]
Thus:
\[ E_{obs} = -p \cdot U_{obs} = -p_0 U^0 = -p_0 \]
So:
\[ \frac{E_r}{E_c} = \frac{R(t_e)}{R(t_r)} = \frac{1}{1 + z} \]
The last step of the algebra producing the original equation is trivial.

5. 12:20

Sol.

(a) Eq. 12.54 gives:
\[ \frac{1}{2} \dot{R}^2 = -\frac{1}{2} k + \frac{4}{3} R^2 (\rho_m + \rho_\Lambda) \]
so, differentiating:

\[
\ddot{R} = \frac{8}{3} R \dot{R} (\rho_m + \rho_\Lambda) + \frac{4}{3} R^2 (\dot{\rho}_m + \dot{\rho}_\Lambda)
\]

\[
\ddot{R} = \frac{8}{3} R \dot{R} (\rho_m + \rho_\Lambda) + \frac{4}{3} R^2 \left[ -3 \rho_0 R_0^3 \frac{\dot{R}}{R^4} \right]
\]

\[
\ddot{R} = \frac{8}{3} R_0 (\rho_0 + \rho_\Lambda) - 4 \rho_0 R_0
\]

\[
\ddot{R} = \frac{4}{3} R_0 (2 \rho_\Lambda - \rho_0)
\]

where the 3rd line came from setting \( R = R_0 \) (today). This is static if \( \rho_0 = -\frac{1}{2} \rho_\Lambda \).

(b) Putting the energy scaling onto the RHS of 12.54 gives a source term of the form:

\[
x = \rho_0 \left[ R_0^3 \frac{R}{R} + \frac{1}{2} R^2 \right]
\]

This does, indeed, differentiate to zero:

\[
x' = \rho_0 \left[ -\frac{R_0^3}{R^2} + 2R \right]
\]

(c) Taking the next derivative:

\[
x'' = \rho_0 \left[ 2 \frac{R_0^3}{R^2} + 2 \right] = 4 \rho_0 > 0
\]

which means that the equilibrium is unstable.