1. [25 points] Short Answers (5 points each)

(a) In a sentence or two, explain why bicycle wheels are large, with all of the mass at the edge, and very light spokes. If you can (and to possibly exceed full credit) explain why wheels on kids bikes are not simply scaled down versions of wheels on adult racing bikes.

**Sol.**

There are a lot of ways to answer this, but all should make some sort of mention of the moment of inertia of a wheel. For fixed mass and radius, $I$ is maximized if the mass is at the edge. High moment of inertia means that it’s hard to tilt. This is also why the radius should be large. The total angular momentum is thus:

$$L = MvR$$

So you get a large angular momentum from large radius, high speed, or large mass.

(up to 2 points EC for discussion below).

In order to reduce the total bike mass, wheels should have a low mass, but you need to ride quickly and have large wheels to make that work.

Kids ride slower than grownups, and the radius is limited by their tiny legs, so the wheels need to be relatively massive to keep their balance.

(b) The Rutherford differential cross-section is given on the equation sheet. As you know, in the original experiment, Rutherford shot helium nuclei (alpha particles) at gold foil at an energy of a few tens of MeV (non-relativistically), and some bounced back.

i. Suppose you double the energy of your alpha particle beam. By what factor will the number of reflected particles increase or decrease?

**Hint:** You do not need to do an integral.

**Sol.**

Note that the differential cross section scales as:

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{E^2}$$

so if you double the energy, the cross section in all directions drops by a factor of 4. So $\times \frac{1}{4}$.
ii. Suppose you double the thickness of the foil. By what factor will the number of reflected particles increase or decrease?

**Sol.**
Doubling the thickness doubles the number of targets, so \( \times 2 \).

(c) Hydrogen consists of an electron orbiting a proton. Positronium consists of a positron and an electron in mutual orbit. Protons and positrons have the same charge, but protons are many, many times the mass of an electron, while positrons have the same mass as an electron.

In quantum mechanics, the energy of the ground state of hydrogen is \(-13.6 \text{ eV}\), but the calculation ultimately reduces to:

\[
E \propto \mu
\]

the reduced mass. Given that, what is the ground state energy of positronium?

**Sol.**
The reduced mass of a hydrogen atom is:

\[
\mu = \frac{m_e m_p}{m_e + m_p} \approx m_e
\]

while for positronium:

\[
\mu = \frac{m_e m_e}{m_e + m_e} = \frac{m_e}{2}
\]

The reduced mass is half as much so the ground state energy is:

\[
E_1 = -6.8 \text{eV}
\]

(d) Noether’s Theorem tells us that each symmetry in the Lagrangian yields a conserved quantity. What is the conserved quantity associated with time symmetry?

**Sol.**
Conservation of energy.

This is known as the Hamiltonian.

**E.C. (1 point)** Give another symmetry-conservation pair.

**Sol.**
Translation → Conservation of Linear Momentum.
Rotation → Conservation of Angular Momentum.

There are others (e.g. charge), but they don’t arise from the Lagrangians discussed in this class.
(e) Two equal mass blocks are suspended over a frictionless pulley, and subject to the constraint:

\[ y_1 + y_2 = \text{const} \]

(which is nothing more than saying that they are connected via a pulley.)

Compute the general Lagrangian (in terms of \( y_1 \) and \( y_2 \) and their time derivatives), and simplify to a 1-d Lagrangian as a function of only \( y_1 \).

**Sol.**

The first version is:

\[
L = \frac{1}{2} m (\dot{y}_1^2 + \dot{y}_2^2) - mg(y_1 + y_2)
\]

Subject to the constraint above:

\[ \dot{y}_1 = -\dot{y}_2 \]

so

\[
L = m\dot{y}_1^2 - \text{const}
\]

2. [15 points] Let’s start with a simple projectile motion problem in 1-d. Consider a mass, \( m \), thrown vertically upwards with an initial speed, \( v_0 \), under the influence of earth-normal gravity, \( g \).

(a) Please express the speed of the ball as a function of height, assuming no air resistance.

**Sol.**

This is simply a matter of energetics:

\[
T = \frac{1}{2} mv^2 = \frac{1}{2} mv_0^2 - mgy
\]
so

\[ v = \sqrt{v_0^2 - 2gy} \]

(b) We almost never actually compute the action of a system:

\[ S = \int L dt \]

but we will today. Consider the action of the particle integrated from \( y = 0 \) to its maximum height. You need not actually compute the value, but determine and justify whether the action is positive, negative, or zero.

**Sol.**

\[ S < 0 \]

Here’s the simplest approach. The potential energy is a linear function of height, which means when \( y = h/2 \), \( L = 0 \). The potential is higher than the kinetic at \( y > h/2 \) (so the Lagrangian is negative), and the potential is lower than the kinetic at \( y < h/2 \) (positive Lagrangian). Which wins?

Since the projectile is moving slower near the top of the arc, it spends more time there, and thus, \( S < 0 \).

More precisely, using the projectile motion equations:

\[
S = \int dt \left[ \frac{1}{2}mv^2 - mgy \right]
\]

\[
= m \int dt_0^{v_0/g} \left[ \frac{1}{2} (v_0 - gt)^2 - m \left( v_0 t - \frac{1}{2} gt^2 \right) \right]
\]

\[
= \frac{-1}{6} \frac{mv_0^3}{g}
\]

which is clearly negative.

(c) We now want to introduce a small quadratic friction of coefficient, \( c \). Assume the friction is small enough that it doesn’t affect the trajectory much. That is, assume \( v(t) \) that you used in part a holds up, and from that, compute the total work done by air resistance on the way up to reaching maximum height.

I promise you that it’s straightforward analytic integral.

**Sol.**
\[
W = \int Fdy \\
= -\int_0^h c v^2 dy \\
= -c \int_0^h (v_0^2 - 2gy) dy \\
= -c [v_0^2 h - \frac{1}{2} gh^2] \\
= -c \left[\frac{v_0^4}{2g} - \frac{v_0^2}{8g}\right] \\
= -\frac{3cv_0^4}{8g}
\]

3. [15 points] Three blocks of identical mass, \( m \), are connected with springs of equal spring constant, \( k \), as shown.

Assume (despite the drawing) that the blocks have negligible width, and that the springs have unstretched length, \( L_0 \).

(a) Write the Lagrangian of the 3 block system. You may ignore gravity and friction.

**Sol.**
This is simply:

\[
\mathcal{L} = \frac{1}{2}m(x_1^2 + x_2^2 + x_3^2) - \frac{1}{2} k \left[(x_3 - x_2 - L_0)^2 + (x_2 - x_1 - L_0)^2\right]
\]

(b) Write the Euler-Lagrange equations for \( x_2 \), the 2nd block.

**Sol.**
First:
\[
\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m\ddot{x}_2
\]

and
\[
\frac{\partial \mathcal{L}}{\partial x_2} = +k(x_3 - x_2 - L_0) - k(x_2 - x_1 - L_0)
\]
which combines to yield:

\[ m\ddot{x}_2 = -k(2x_2 - x_1 - x_3) \]

(c) Suppose that the two outer masses are held fixed: \(x_3 = +L, x_1 = -L\), and the middle mass is given a small kick. Describe its subsequent motion, including the frequency of oscillation, if relevant.

**Sol.**
In this case, \(x_1 + x_3 = 0\), so:

\[ m\ddot{x}_2 = -2kx_2 \]

which is oscillatory motion. The frequency of oscillation is:

\[ \omega = \sqrt{\frac{2k}{m}} \]
4. [20 points] Consider the orbital dynamics around an earth which (contrary to reality) is perfectly smooth, spherical, and completely devoid of mountains or ravines.

(a) To get us started, imagine launching a particle around the equator, just above the surface of the earth. What is the angular frequency of the orbit in terms of the mass of the earth, $M_\oplus$, the radius, $R_\oplus$, and physical constants?

**Sol.**
First, note that:

$$v = R \omega = \sqrt{\frac{GM}{R}}$$

so

$$\omega = \sqrt{\frac{GM}{R^3}}$$

Note that you could also have used Kepler’s 3rd law to solve this.

(b) Now suppose you want to launch a rocket from our smooth earth to the moon. For convenience, let’s assume that neither the mass nor radius of the moon are significant, but that the moon is $49R_\oplus$ from the center of the earth. Assuming the rocket blasts off horizontally at perigee (closest approach to earth) and hits the moon at apogee (furthest approach), as shown in the right panel, what is the eccentricity and semi-major axis of the orbit?

Put more simply: $r_P = 1R_\oplus$, $r_A = 49R_\oplus$, what is $a$ and $\epsilon$?

**Sol.**
As we’ve seen:

$$r_P + r_A = 2a = 50R_\oplus$$

so

$$a = 25R_\oplus$$

From the definitions:

$$\frac{r_P}{a} = \frac{1 - \epsilon^2}{1 + \epsilon} = 1 - \epsilon = \frac{1}{25} = 0.04$$
so

\[ \epsilon = 0.96 \]

(c) Now we’ll mix it up a bit. I assert that if the earth had a constant density (which it doesn’t) then the gravitational potential in the interior of the earth could be written as:

\[ U(r) = \frac{GM\oplus m r^2}{2R_\oplus^3} \]

Now assume we burrow a small hole through the center of the earth, and dropped a particle into the hole. What would the force on the particle be as a function from distance to the center?

**Sol.**

We know:

\[ \vec{F} = -\nabla U = -\frac{GM mr}{R^3} \hat{r} \]

(d) If you let the mass drop through the earth, it will eventually reach the other side, turn around, and come back to where you started. What is the angular frequency of the mass oscillating through the middle of the earth?

I’d strongly urge you to note any similarities between the force you determined in part (c) and those that we’ve studied in this class.

**Sol.**

It is simplest to note that this system is just like a SHO with spring constant:

\[ k = \frac{GMm}{R^3} \]

so

\[ \omega = \sqrt{\frac{GM}{R^3}} \]

The same as for the orbit around the planet.

(e) **E.C. (2 points)** How would your answer to the previous part change if the mass could travel through any trajectory interior to the earth, not just a straight path through the center?

**Sol.**

It wouldn’t. The force still acts like a 2d SHO, and we’ve found from class that the frequency is just a function of the spring constant.
5. [25 points] Consider a small cube of mass, \( M \), and length, \( L \), on a side. It is tilted an amount \( \theta \) on one edge without slipping, as shown:

Because I care so very deeply about you, I’m going to just give you the Lagrangian:

\[
\mathcal{L} = \frac{1}{4}ML^2\dot{\theta}^2 - \frac{MgL}{2} (\sin \theta + \cos \theta - 1)
\]

(Note that this Lagrangian only holds between \( 0 \leq \theta \leq \pi/2 \)).

(a) What is the canonical momentum, \( p_\theta \), of the tilted cube?

**Sol.**

The canonical momentum is simply:

\[
p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{1}{2}ML^2\dot{\theta}
\]

We’ll also probably need the inverse later:

\[
\dot{\theta} = \frac{2p_\theta}{ML^2}
\]

(b) What is the Hamiltonian of the tilted cube, expressed as a function of \( \theta \), \( p_\theta \), and constants only?

**Sol.**

The Hamiltonian may be expressed as:

\[
H = p_\theta \dot{\theta} - \mathcal{L}
\]

\[
= \frac{1}{2}ML^2\dot{\theta}^2 - \left[ \frac{1}{4}ML^2\dot{\theta}^2 - \frac{MgL}{2} (\sin \theta + \cos \theta - 1) \right]
\]

\[
= \frac{1}{4}ML^2\dot{\theta}^2 + \frac{MgL}{2} (\sin \theta + \cos \theta - 1)
\]

\[
= \frac{4p_\theta^2}{ML^2} + \frac{MgL}{2} (\sin \theta + \cos \theta - 1)
\]
(c) Please write down Hamilton’s equations for the system.

**Sol.**

\[
\frac{\partial H}{\partial \theta} = -\dot{p}_\theta = \frac{MgL}{2} (\cos \theta - \sin \theta)\\
\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{2p_\theta}{ML^2}
\]

The latter of which, of course, we already knew.

(d) Looking at Hamilton’s equations, can you find any equilibria? That is, for which value of \(\theta\) is \(\dot{p}_\theta = 0\)?

Is the equilibrium stable or unstable? Be sure to explain your reasoning.

**Sol.**

First note:

\[
\dot{p}_\theta = \frac{MgL}{2} (\sin \theta - \cos \theta)
\]

The only equilibrium can be if the term in the parenthesis cancels, which is \(\theta = \pi/4\).

Is it stable?

There are a number of approaches, but perhaps the simplest is to take the second derivative of the potential:

\[
U = \frac{mgL}{2} (\sin \theta + \cos \theta - 1)\\
U' = \frac{mgL}{2} (\cos \theta - \sin \theta)\\
U'' = \frac{mgL}{2} (-\sin \theta - \cos \theta)
\]

As the second derivative is always negative, the equilibrium is unstable.

Of course, reasoning about the physical system would tell you the same thing. Balance a cube on the edge, and it will quickly topple.

(e) Consider a situation in which the cube is initialized with \(\theta = \pi/6\), \(p_\theta = 0\) (more or less as drawn above). Sketch a phase-space diagram, and draw a point representing the initial conditions.

Add a short line-segment (with an arrow, indicating direction), showing the state of the system a short time later.

**Sol.**