HW 3 Solution Key

1. (10 points) Suppose a system has a potential energy:

\[ U = A(x^2 - R^2)^2 \]

where \( A \) is a constant and \( R \) is some fixed distance from the origin (on the positive and negative side).

(a) Sketch the potential energy, and identify all equilibria. Are they stable or unstable.

**Note:** Please try to sketch it over a range in \( x \) such that all equilibria are clearly visible.

**Sol.**

This is what is known as the “Higgs Potential” and is very important in particle physics.

This has two minima, at \( x = \pm R \), and a local maximum, \( x = 0 \). The first two are stable equilibria, and the latter is an unstable equilibrium.

(b) Consider a particle at a position \( x = R + dr \) (where \( dr \) is assumed to be small). What is the force on the particle? Be sure to include the direction.

**Sol.**

First, let’s compute the force in general:

\[ F = -\frac{dU}{dx} = -4Ax(x^2 - R^2) = -4A(x^3 - xR^2) \]

For small values of \( dr \):

\[ (R + dr)^3 \approx R^3 + 3R^2dr + \mathcal{O}(dr^2) \]

and

\[ R^2x = R^2(R + dr) = R^3 + R^2dr \]

so

\[ F = -(8AR^2)dr \]

where the term in the parentheses is a spring constant. Note that the force is opposite the direction of the displacement, and is linear. Hence, a spring!

(c) Suppose a particle is released from rest at \( x = R + dr \) at \( t=0 \). Describe the subsequent motion \( x(t) \), in terms of \( R \), \( A \), and \( dr \).

**Sol.**

It oscillates like a spring, with a frequency:

\[ \omega = \sqrt{\frac{8AR^2}{m}} \]

so

\[ x(t) = R + dr \cos(\omega t) \]
2. 5.2 (15 points)

The potential energy of two atoms in a molecule is approximated by:

\[ U(r) = A \left[ \left( e^{(R-r)/S} - 1 \right)^2 - 1 \right] \]

(a) Sketch the function.

**Sol.**

Note a couple of things. First, if \( R \) is large, then \( e^{R/S} \to \text{big} \). However, no matter what the exponential is larger than 0, so the term in the inner parentheses is at smallest, -1, so the square turns around at 1, and thus the overall term turns around at \( r = R \), \( U(r) = -A \). As \( r \to \infty \), the exponential goes to 0, so the overall expression goes to 0.

Thus:

(b) Find the separation, \( r_0 \) at which \( U(r) \) is minimum.

**Sol.**

As determined, it’s \( r = R \).

(c) Now write \( r = R + x \) so that \( x \) is the displacement from equilibrium. Compute the spring constant.

**Sol.**

First, note that: \( R - r = -x \). For small values of \( x/S \):

\[ e^{-x/S} \simeq 1 - \frac{x}{S} \]

so

\[ e^{-x/S} - 1 \simeq -\frac{x}{S} \]

and thus:

\[ (e^{-x/S} - 1)^2 - 1 = \frac{x^2}{S^2} - 1 \]

or

\[ \frac{1}{2} k x^2 = \frac{A}{S^2} x^2 \]

so:

\[ k = \frac{A}{S^2} \]

3. 5.17 (10 points)

Consider the two-dimensional anisotropic oscillation.

(a) Prove that if the ratio of frequencies is rational then the motion is periodic. What is the period?

**Sol.**
A system will return to its initial state in the x-direction when:

$$\omega_x t = 2\pi N$$

and similarly for y. For the system to reset:

$$NP_x = MP_y$$

for integer values, or equivalently:

$$\frac{N}{M} = \frac{P_y}{P_x} = \frac{\omega_x}{\omega_y}$$

for integer values of N, M. This is the definition of a rational ratio.

The period is $$NP_x$$, where N and M are reduced to the lowest common denominator.

(b) **Prove that if the same ratio is irrational, the motion never repeats itself.**

The previous result shows that a rational ratio is both necessary and sufficient. Frankly, I’m not sure why Taylor made this two separate problems.

4. 6.2 (10 points) Do the same as in Problem 6.1 (setting up the path length) but find the length L of a path on a cylinder of radius, R, using cylindrical polar coordinates. Assume the path is specified by the form, $$\phi = \phi(z)$$.

**Sol.**

First, note that in cylindrical coordinates, a small distance may be specified by:

$$dl^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

but with fixed $$\rho = R$$, we get:

$$dl^2 = R^2 d\phi^2 + dz^2$$

so

$$L = \int \sqrt{R^2 d\phi^2 + dz^2}$$

$$= \int dz \sqrt{1 + R^2 \left( \frac{d\phi}{dz} \right)^2}$$

5. 6.7 (Closely related to 6.2) (15 points)

Compute the Euler-Lagrange of 6.2 and solve. Is it unique?

**Sol.**

First, take the Euler-Lagrange equations for

$$f(\phi, \phi', z) = \sqrt{1 + R^2 (\phi')^2}$$

so

$$\frac{\partial f}{\partial \phi} = \frac{\phi'}{\sqrt{1 + R^2 (\phi')^2}}$$

and

$$\frac{\partial f}{\partial \phi'} = 0$$

so

$$\frac{\phi'}{\sqrt{1 + R^2 (\phi')^2}} = C$$
which is solved for $\phi' = \text{const}$.

If you unravel a cylinder, this produces a straight line.

Note that this is not unique. Consider, for example, the case where $(z_1, \phi_1) = (0, 0)$, and $(z_2, \phi_2) = (h, 0)$. There are an infinite number of solutions which take the form:

$$\phi = 2\pi n \frac{z}{h}$$

for any integer, positive or negative.

6. 6.11 (15 points) Find the solution for which

$$\int \sqrt{x} \sqrt{1 + (y')^2} \, dx$$

is stationary.

Sol.

Writing the Euler-Lagrange equations:

$$\frac{\partial f}{\partial y'} = \sqrt{x} \frac{y'}{\sqrt{1 + (y')^2}}$$

and

$$\frac{\partial f}{\partial y} = 0$$

so

$$\sqrt{x} \frac{y'}{\sqrt{1 + (y')^2}} = C$$

$$x \frac{(y')^2}{1 + (y')^2} = C^2$$

$$x(y')^2 = C^2 + C^2 (y')^2$$

$$(y')^2 = \frac{C^2}{x - C^2}$$

$$y' = \frac{C}{\sqrt{x - C^2}}$$

Solving:

$$y = 2C \sqrt{x - C^2}$$

7. 6.17 (10 points) Find the equations for a geodesic on a cone.

Sol.

In cylindrical coordinates, the radius and height can be related:

$$z = \lambda \rho$$

and thus:

$$dl^2 = dz^2 + d\rho^2 + \rho^2 d\phi^2$$

$$= (1 + \lambda^2) d\rho^2 + \rho^2 d\phi^2$$

and thus:

$$L = \int d\rho \sqrt{(1 + \lambda^2) + \rho^2 (\phi')^2}$$
where $\phi' = d\phi/d\rho$.

The Euler-Lagrange equations are thus:

$$\frac{\partial f}{\partial \phi'} = \frac{\rho^2 \phi'}{\sqrt{1 + \lambda^2} + \rho^2 (\phi')^2}$$

and

$$\frac{\partial f}{\partial \phi} = 0$$

which yields:

$$\frac{\rho^2 \phi'}{\sqrt{1 + \lambda^2} + \rho^2 (\phi')^2} = C$$

or

$$\rho^4 (\phi')^2 = C^2 \left[ (1 + \lambda^2) + \rho^2 (\phi')^2 \right]$$

$$\left[ \rho^4 - C^2 \rho^2 \right] (\phi')^2 = C^2 (1 + \lambda^2)$$

$$\phi' = \frac{C \sqrt{1 + \lambda^2}}{\sqrt{\rho^4 - C^2 \rho^2}}$$

8. 6.22 – This is a toughy, but if you understand it, you’ll be in good shape. (15 points)

You have a string of fixed length which is begins at the origin, and terminates at the x-axis. Show that the maximum area is contained with a semi-circle.

Sol.

First, consider the area contained in a string:

$$A = \int y(x)dx$$

If the string has length, $L$, then a better variable of integration is $L = \int dl$. Note that:

$$dl^2 = dx^2 + dy^2$$

so

$$dx = dl \sqrt{1 - (y')^2}$$

where

$$y' = \frac{dy}{dl}$$

so the area may be expressed as:

$$A = \int dl \ y \sqrt{1 - (y')^2}$$

so

$$\frac{\partial f}{\partial y'} = -\frac{yy'}{\sqrt{1 - (y')^2}}$$

Problem 6.20 shows that:

$$f - y' \frac{\partial f}{\partial y'} = C$$

so:

$$y \sqrt{1 - (y')^2} + \frac{y(y')^2}{\sqrt{1 - (y')^2}} = C$$

(1)

Now, if the solution is a semi-circle, then:

$$\theta = \frac{l}{R}$$
and

\[ y = R \sin \theta = R \sin \left( \frac{l}{R} \right) \]

so

\[ y' = \cos \left( \frac{l}{R} \right) \]

The first term in equation (1) above yields:

\[ y \sqrt{1 - (y')^2} = R \sin^2(\theta) \]

and the second term:

\[ \frac{y(y')^2}{\sqrt{1 - (y')^2}} = R \cos^2 \theta \]

The two terms add to a constant, \( R \).

Thus, the semi-circle does indeed maximize the area.